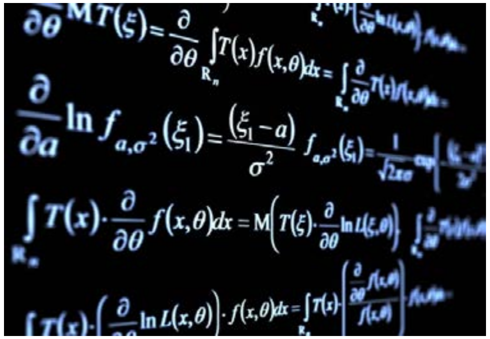

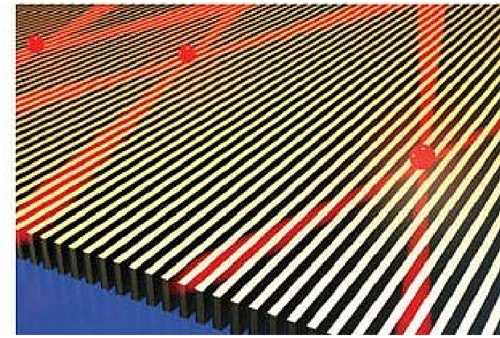

An Introduction on Periodic Structures in Electromagnetics

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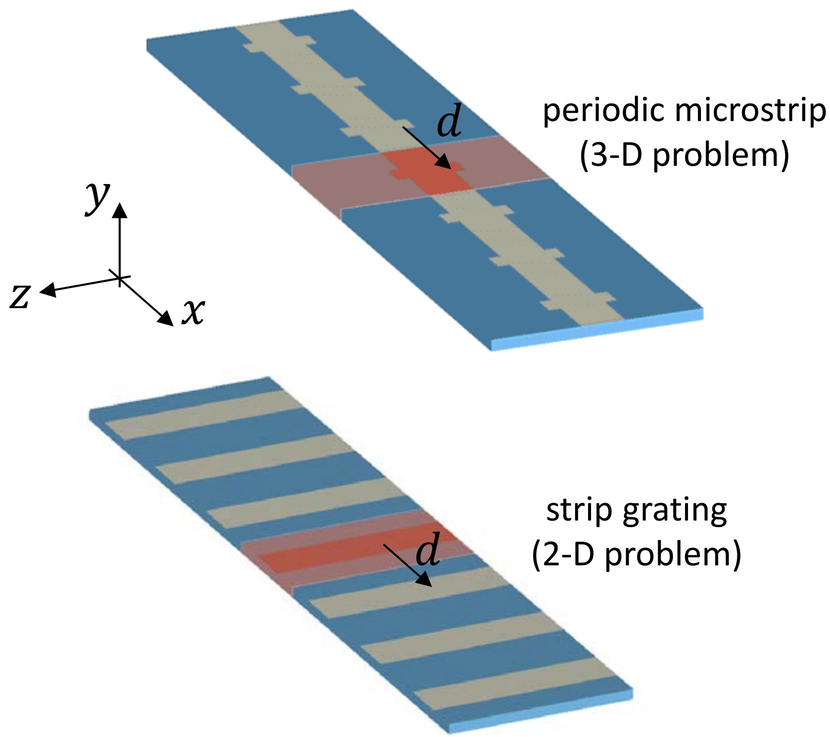
1

Periodic Structures and Bloch Modes

1-D Periodic Structures

A structure is periodic if its geometry and physical properties are invariant under a translation:



$$T_{d\hat{\mathbf{x}}}: \begin{cases} x \to x + d \\ y \to y \\ z \to z \end{cases}$$

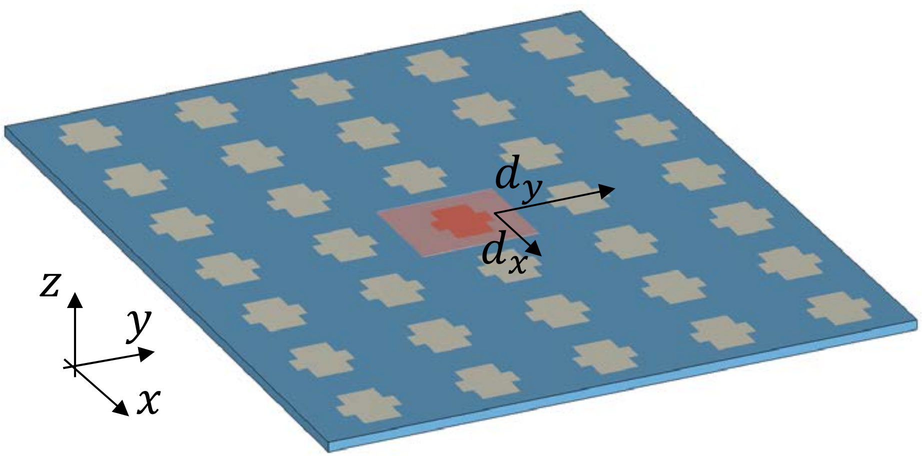
Translation along x

If we have invariance under a translation of p, we have it also under np, n integer

The minimal p is called the "period" of the structure.
The restriction of the structure to a period is the **unit cell** U

2-D Periodic Structures

A structure is periodic if its geometry and physical properties are invariant under a translation:



$$T_{d_x\hat{\mathbf{x}}+d_y\hat{\mathbf{y}}}: \begin{cases} x \to x + d_x \\ y \to y + d_y \\ z \to z \end{cases}$$

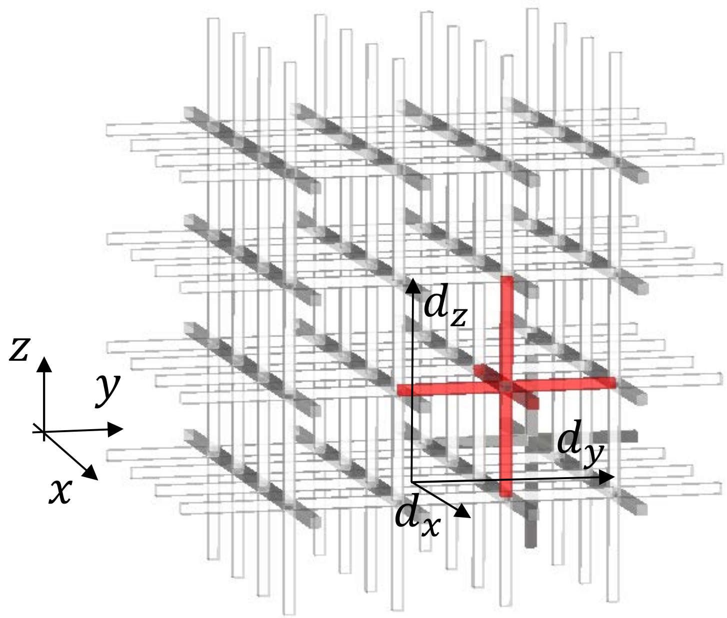
Translation along x and y

If we have invariance under a translation of p_i, we have it also under np_i, n integer

The minimal p_i is called the “period” of the structure.
The restriction of the structure to a period is the **unit cell** U

3-D Periodic Structures

A structure is periodic if its geometry and physical properties are invariant under a translation:



$$T_{d_x\hat{\boldsymbol{x}}+d_y\hat{\boldsymbol{y}}+d_z\hat{\boldsymbol{z}}}: \begin{cases} x \to x + d_x \\ y \to y + d_y \\ z \to z + d_z \end{cases}$$

Translation along x, y, z

If we have invariance under a translation of p_i, we have it also under np_i, n integer

The minimal p_i is called the “period” of the structure.
The restriction of the structure to a period is the **unit cell** U

Periodic Structures and Eigenproblems

When a structure is invariant under an operator $\boldsymbol{A}$, its modes are solutions of the eigenproblem

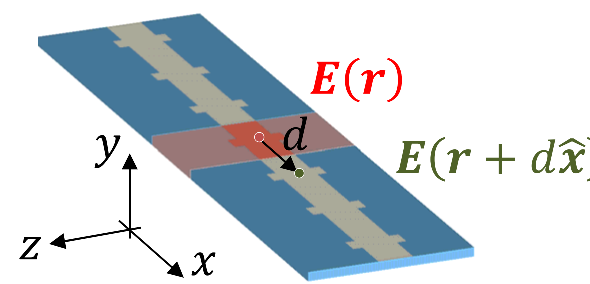
$$\boldsymbol{A}[\boldsymbol{E}(\boldsymbol{r})] = \lambda\, \boldsymbol{E}(\boldsymbol{r})$$

where $\boldsymbol{A}$ is an operator acting on a field $\boldsymbol{E}$, and λ is a scalar factor (not depending on $\boldsymbol{r}$).

A non-trivial solution $\boldsymbol{E}$ is an *eigenvector* of the problem, and λ is an *eigenvalue*.

In periodic structure, $\boldsymbol{A}$ is a translation, and modes are called "Floquet-Bloch modes" or Bloch modes. In 1-D periodicity:

$$T_{d\hat{\boldsymbol{x}}}[\boldsymbol{E}(\boldsymbol{r})] = \boldsymbol{E}(\boldsymbol{r} + d\hat{\boldsymbol{x}}) = \lambda\, \boldsymbol{E}(\boldsymbol{r})$$



Bloch modes do not change after a period, apart from a multiplicative constant λ

Floquet-Bloch Modes

$$T_{d\hat{\boldsymbol{x}}}[\boldsymbol{E}(\boldsymbol{r})] = \boldsymbol{E}(\boldsymbol{r} + d\hat{\boldsymbol{x}}) = \lambda \boldsymbol{E}(\boldsymbol{r})$$

We call $\lambda = e^{-jk_x d} = e^{-\alpha_x d} e^{-j\beta_x d}$

where $k_x = \beta_x - j\alpha_x$

Attenuation constant

Phase constant

Propagation constant

$$\boldsymbol{E}(\boldsymbol{r} + d\hat{\boldsymbol{x}}) = e^{-jk_x d}\boldsymbol{E}(\boldsymbol{r}) = e^{-\alpha_x d} e^{-j\beta_x d}\boldsymbol{E}(\boldsymbol{r})$$

Moving the observation point of a period, the field is

- phased of $\beta_x d$
- attenuated of $\alpha_x d$

Note: the field is *not* periodic!

Floquet-Bloch Modes and Propagation

Let us define the function $u(x,y,z) = e^{jk_x x} E(x,y,z)$

How does the u function behaves if translated of a period?

$$u(x+d,y,z) = e^{jk_x(x+d)} E(x+d,y,z) = e^{jk_x x} e^{jk_x d} E(x+d,y,z) =$$

$$= e^{jk_x x}\, e^{jk_x d} e^{-jk_x d} E(x,y,z) = e^{jk_x x}\, E(x,y,z) = u(x,y,z)$$

So the u function is periodic, and $E(x,y,z) = e^{-jk_x x}\, u(x,y,z)$

The field is a product of a periodic function (describing the field variation inside a unit cell) and a propagation factor $e^{-jk_x x}$

The same can be done in 2-D and 3-D periodic structures…

Bloch Modes and Spatial Harmonics

The periodic function $u(x,y,z)$ is $L^2(U)$ (a finite energy is associated to the field inside a unit cell *U*) and can be written as a Fourier series

$$E(x,y,z) = e^{-jk_x x} u(x,y,z) = e^{-jk_x x} \sum_{n=-\infty}^{+\infty} c_n(y,z) e^{-j\frac{2\pi n}{d}x} = \sum_{n=-\infty}^{+\infty} c_n(y,z) e^{-j\left(k_x + \frac{2\pi n}{d}\right)x}$$

Each term of the series is a **spatial harmonic:** a plane wave propagating with a wavenumber

$$k_{x,n} = k_x + \frac{2\pi n}{d} = \beta_x + \frac{2\pi n}{d} - j\alpha_x$$

- The phase constant of the nth harmonic is $\beta_x + \frac{2\pi n}{d}$
- The attenuation constant α_x is the same for all the harmonics.

The weights $c_n(y,z)$ of each harmonic depend on the cell configuration.
In general, all the harmonics are present and they cannot be excited separately.
The only field which can be excited is the entire Bloch mode.

This harmonic sum can be easily generalized in 2-D or 3-D periodic structures.

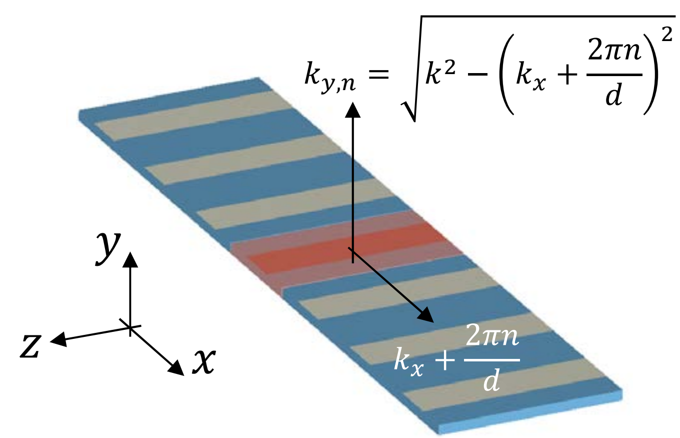
Spatial Harmonics in 2-D Geometries

$$E(x,y,z) = \sum_{n=-\infty}^{+\infty} c_n(y,z)e^{-j\left(k_x+\frac{2\pi n}{d}\right)x}$$

In free space (homogeneous and isotropic medium), each Cartesian component of E must satisfy the Helmholtz equation $\nabla^2 E + k^2 E = 0$

This means that, e.g. in a strip grating, $k_y = \sqrt{k^2 - \left(k_x + \frac{2\pi n}{d}\right)^2}$:

$$E(x,y) = \sum_{n=-\infty}^{+\infty} c_n e^{-j\sqrt{k^2-\left(k_x+\frac{2\pi n}{d}\right)^2}\,y} e^{-j\left(k_x+\frac{2\pi n}{d}\right)x}$$



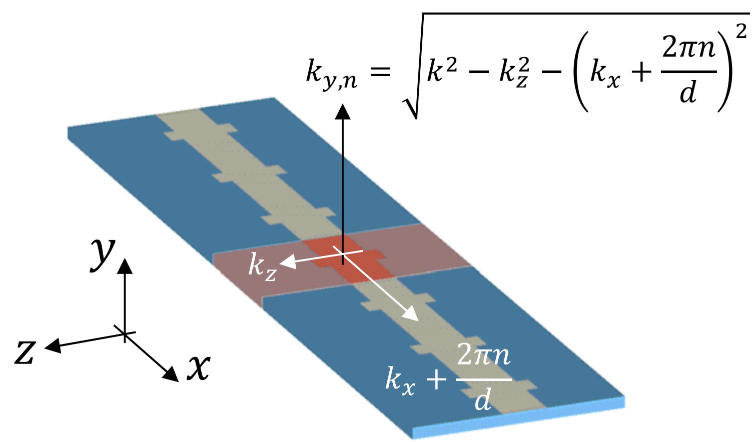
Spatial Harmonics in 3-D Geometries

$$E(x,y,z) = \sum_{n=-\infty}^{+\infty} c_n(y,z) e^{-j\left(k_x+\frac{2\pi n}{d}\right)x}$$

If the 1-D periodic line has a dependence on three variable (microstrip line), an integral over a continuous wavenumber must be added.

$$E(x,y,z) = \frac{1}{2\pi d} \sum_{n=-\infty}^{+\infty} e^{-j\left(k_x+\frac{2\pi n}{d}\right)x} \int_{-\infty}^{+\infty} c_n(k_z) e^{-j\sqrt{k^2-k_z^2-\left(k_x+\frac{2\pi n}{d}\right)^2}\,y} e^{-jk_z z} \mathrm{d}k_z$$

The k_z integral can be interpreted as the Fourier transform of $c_n(y,z)$ with respect to z



2

Formulation of Periodic Problems and Dispersive Equations

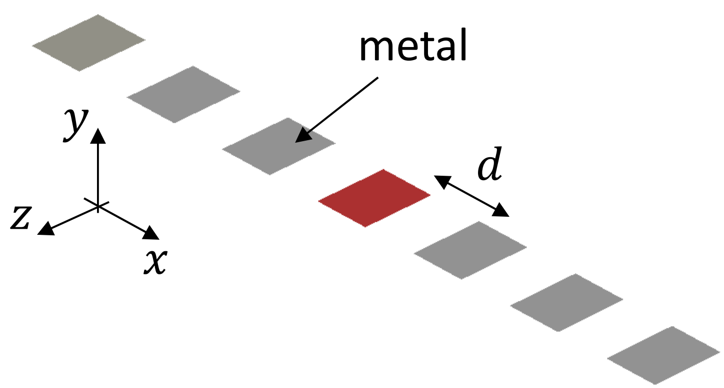
Formulating a Periodic Problem

How to formulate a dispersive periodic problem?

Dispersive problem: no sources are present (we look for Bloch modes).

Remember: in a scattering problem (with sources *having the same periodicity of the structure*) a set of these Bloch modes will be excited.

Let our structure be composed of a 1-D periodic pattern of metallic elements.
We need to enforce boundary conditions on each element to determine the solution.

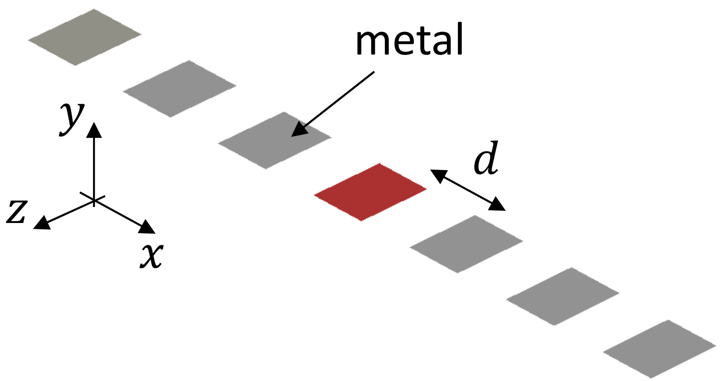


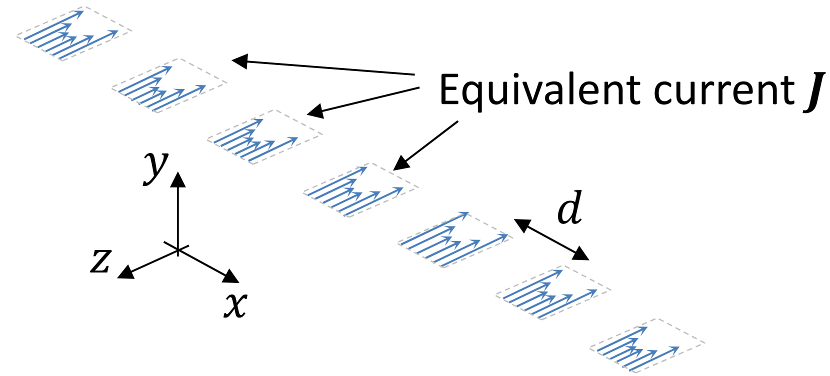
$$\boldsymbol{E}^{\tan}(\boldsymbol{r}) = \boldsymbol{0}, \quad \boldsymbol{r} \text{ on each metallic element}$$

Floquet periodicity

$$\boldsymbol{E}^{\tan}(\boldsymbol{r}) = \boldsymbol{0}, \quad \boldsymbol{r} \text{ on the } n = 0 \text{ metallic element}$$

Formulating a Periodic Problem: the Equivalence Theorem





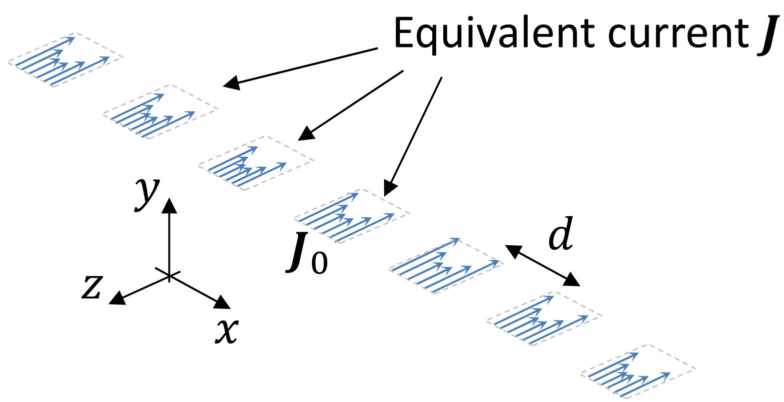
Equivalent currents on each element can be regarded as sources of the Bloch mode.
They replace the metallic elements (equivalence theorem) and radiate $\boldsymbol{E}$ in free space.

The Bloch mode can therefore be expressed with a Green's function formalism.

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{all elements}} \underline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}'; \omega) \cdot \boldsymbol{J}(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}'$$

Free-space electric-field/electric-source Green's function

Formulating a Periodic Problem inside a Unit Cell



$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{all elements}} \underline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}'; \omega) \cdot \boldsymbol{J}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

Convolution integral

The Bloch mode (and then the current $\boldsymbol{J}$) is Floquet-periodic:

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=-\infty}^{+\infty} e^{-jnk_x d} \boldsymbol{J}_0(\boldsymbol{r} - nd\hat{\boldsymbol{x}})$$

$\boldsymbol{J}_0$ being the current on one unit cell

Formulating a Periodic Problem inside a Unit Cell

This simplifies the convolution integral:

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{all elements}} \underline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}';\omega) \cdot \sum_{n=-\infty}^{+\infty} e^{-jnk_x d} \boldsymbol{J}_0(\boldsymbol{r}' - nd\widehat{\boldsymbol{x}})\, \mathrm{d}\boldsymbol{r}'$$

$$= \sum_{n=-\infty}^{+\infty} e^{-jnk_x d} \int_{\text{element } 0} \underline{\boldsymbol{G}}(\boldsymbol{r} - nd\widehat{\boldsymbol{x}},\boldsymbol{r}';\omega) \cdot \boldsymbol{J}_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' =$$

$$= \int_{\text{elem. } 0} \sum_{n=-\infty}^{+\infty} e^{-jnk_x d}\, \underline{\boldsymbol{G}}(\boldsymbol{r} - nd\widehat{\boldsymbol{x}},\boldsymbol{r}';\omega) \cdot \boldsymbol{J}_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' = \int_{\text{elem. } 0} \underline{\boldsymbol{G}}^{\boldsymbol{p}}(\boldsymbol{r},\boldsymbol{r}';\omega,k_x) \cdot \boldsymbol{J}_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'$$

where $\underline{\boldsymbol{G}}^{\boldsymbol{p}}(\boldsymbol{r},\boldsymbol{r}',k_x)=\sum_{n=-\infty}^{+\infty} e^{-jnk_x d}\, \underline{\boldsymbol{G}}(\boldsymbol{r} - nd\widehat{\boldsymbol{x}},\boldsymbol{r}')$
is a **periodic Green's function**, keeping all the information about periodicity.

The boundary conditions can be therefore enforced on a single unit cell.

Periodic Green's Function and Integral Equations

The boundary conditions is formulated as an integral equation in the unknown $\boldsymbol{J}$:

$$\left. \int_{\text{element } 0} \underline{\boldsymbol{G}}^{p}(\boldsymbol{r}, \boldsymbol{r}', \omega, k_x) \cdot \boldsymbol{J}(\boldsymbol{r}') \mathrm{d}\boldsymbol{r}' \right|_{\tan} = \boldsymbol{0} \qquad \boldsymbol{r} \text{ in element } 0$$

This is called an Electric-Field Integral equation (EFIE).

Other kinds of Integral equations can be formulated according to the kind of boundary condition to enforce, type of objects, etc.

In a periodic problem the analysis can be restricted to a single unit cell by

- using a periodic Green's function (in a MoM formulation),
- enforcing explicitly the Floquet-conditions on opposite sides of a unit cell (FEM, FDTD).

The second approach is used by commercial software like CST Microwave Studio, Ansys HFSS...

Discretization and Dispersion Equation

$$\int_{\text{element } 0} \underline{\boldsymbol{G}}^p(\boldsymbol{r}, \boldsymbol{r}', \omega, k_x) \cdot \boldsymbol{J}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \Bigg|_{\tan} = \boldsymbol{0} \qquad \boldsymbol{r} \text{ in element } 0$$

A numerical solution is found with the *Method of Moments*, MoM (implemented e.g. in FEKO). The MoM converts the integral equation into a linear algebraic system:

$$\underline{\boldsymbol{M}}(\omega, k_x) \cdot \boldsymbol{I} = \boldsymbol{0}$$

The dispersive problem is solved by looking for non-trivial solutions $\boldsymbol{I}$

They exist only if the determinant of the $\underline{\boldsymbol{M}}$ matrix is zero:

$$\det\left[\underline{\boldsymbol{M}}(\omega, k_x)\right] = 0$$ "Dispersion equation" of the structure

It is solved for each frequency ω and gives the relation $k_x = k_x(\omega)$ of the periodic structure.

Note that commercial software can solve the problem for each β and find the function $\omega=\omega(\beta_x)$.

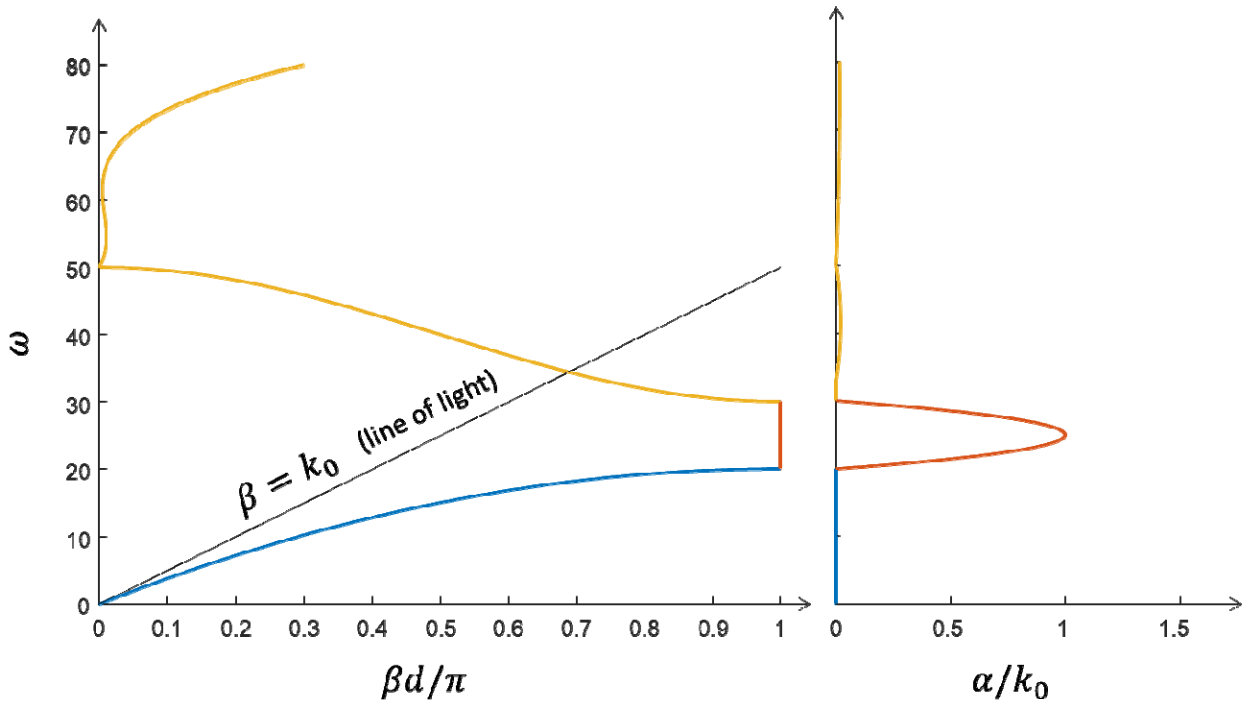
3

The Brillouin Diagram and Propagation Regimes

The Brillouin Diagram in 1-D

$$\det\left[\underline{\boldsymbol{M}}(\omega, k_x)\right] = 0 \quad \Rightarrow \quad k_x = k_x(\omega)$$

We can visualize this dispersive equation with the help of the “Brillouin diagram”.



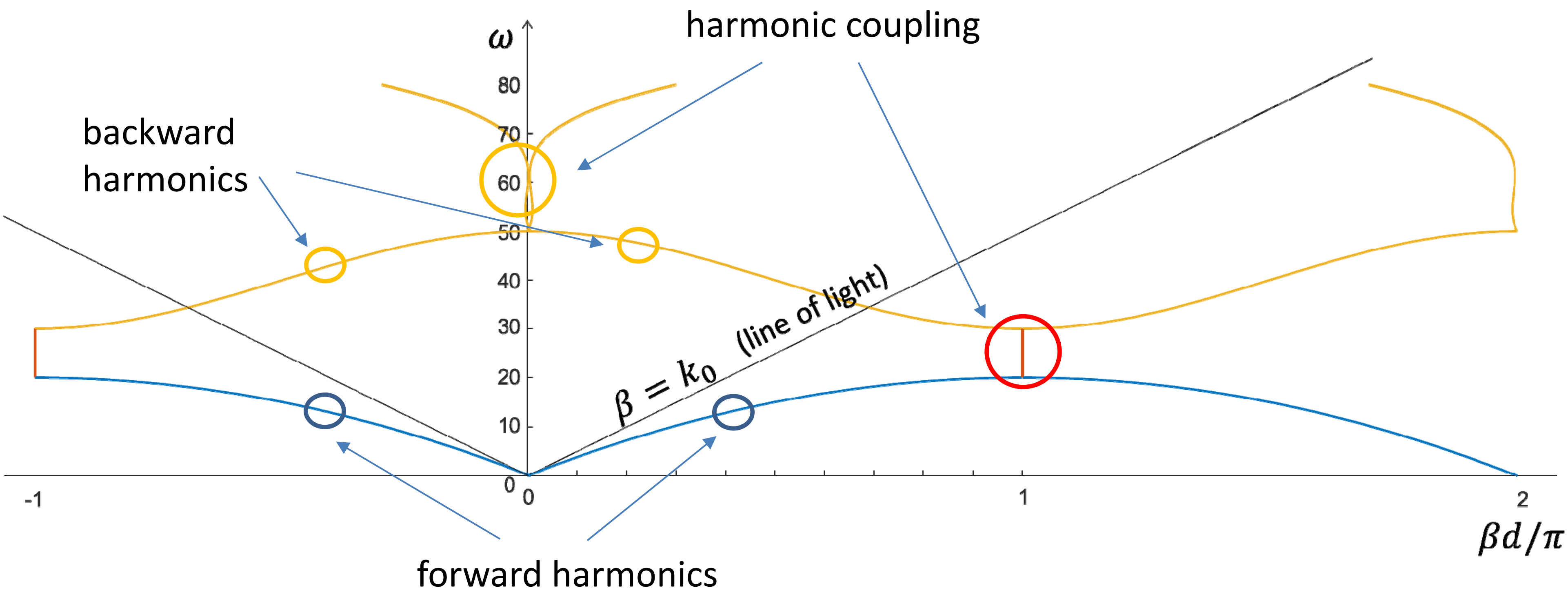
We show only the section $0 < \beta d < \pi$ since the diagram is

periodic of 2π (the βd of the spatial harmonics differ of 2π)

symmetric $\beta \to -\beta$ due to reciprocity

The same visualization is used in solid-state physics (energy bands vs. reciprocal lattice).

The “Complete” Brillouin Diagram in 1-D



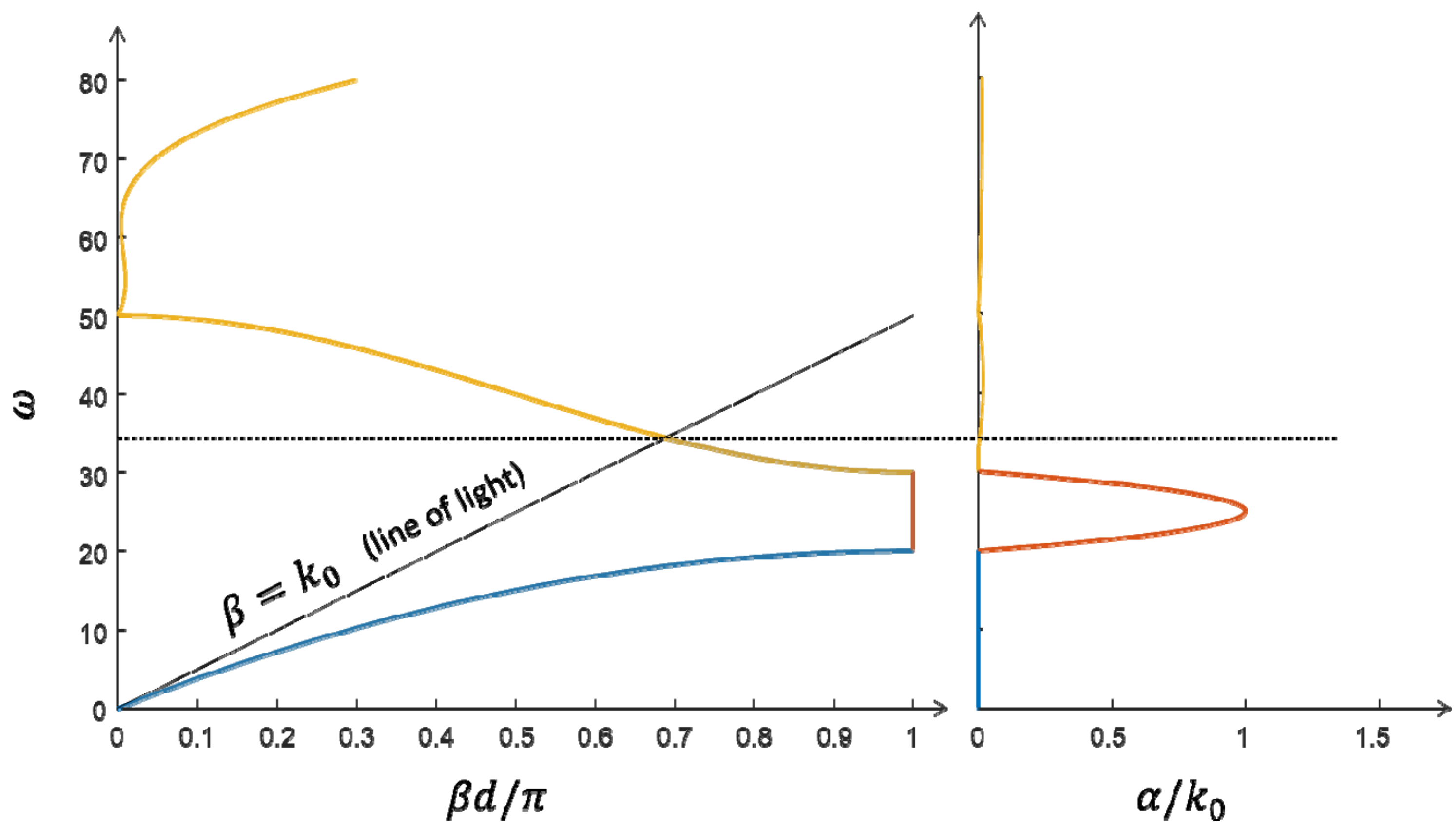
- Brilouin diagram gives no information on the excitation of the harmonics!
- If the diagram is plotted with a doubled period, one harmonics every two is absent.
- And what if the structure is uniform?

Bound Regimes

A slow wave ($\beta > k_0$) has an imaginary $k_y = \sqrt{k^2 - \beta^2}$

The wave attenuates far from the structure.

A slow wave is a **bound wave**.

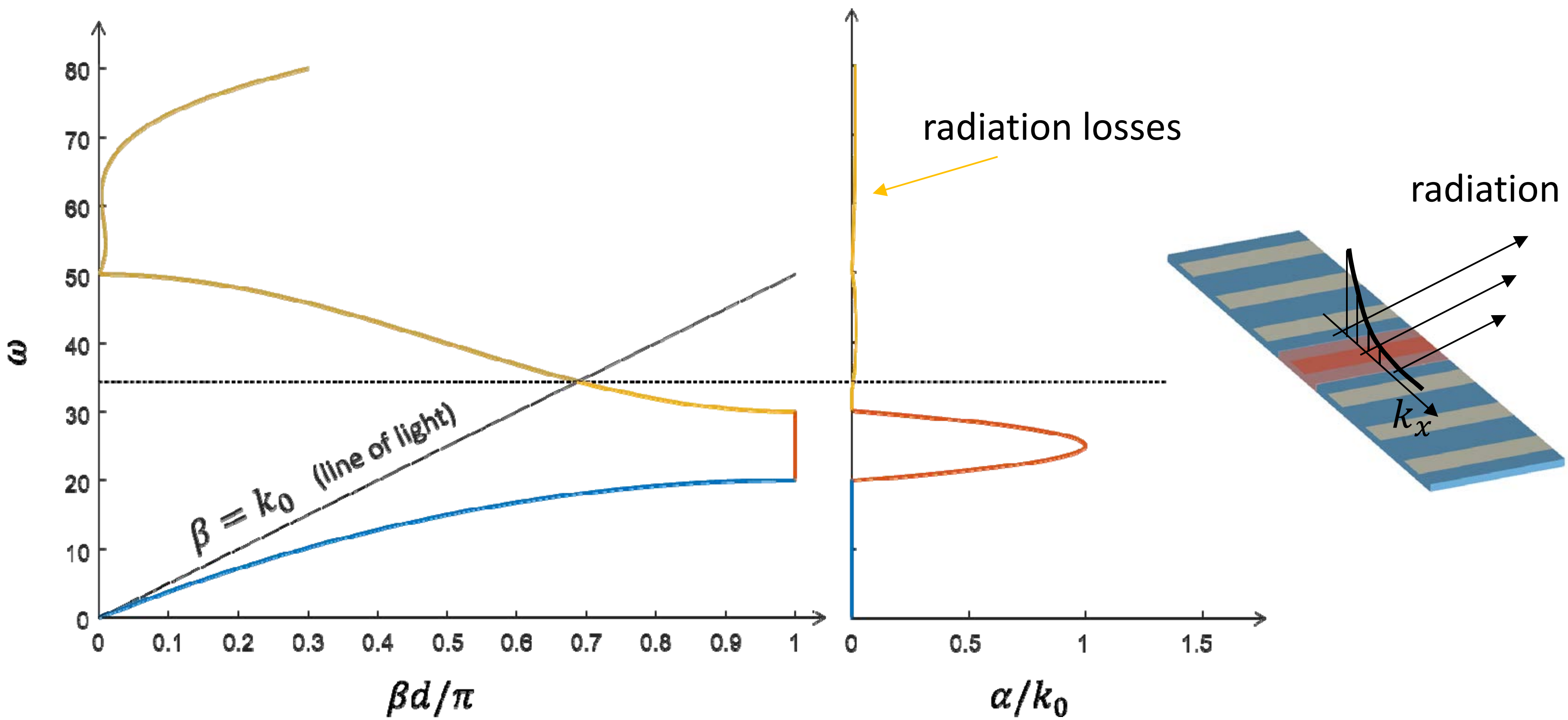


Radiating or Leaky Regimes

A fast wave ($\beta < k_0$) has $k_y = \sqrt{k^2 - \beta^2}$ with a real part different from zero.

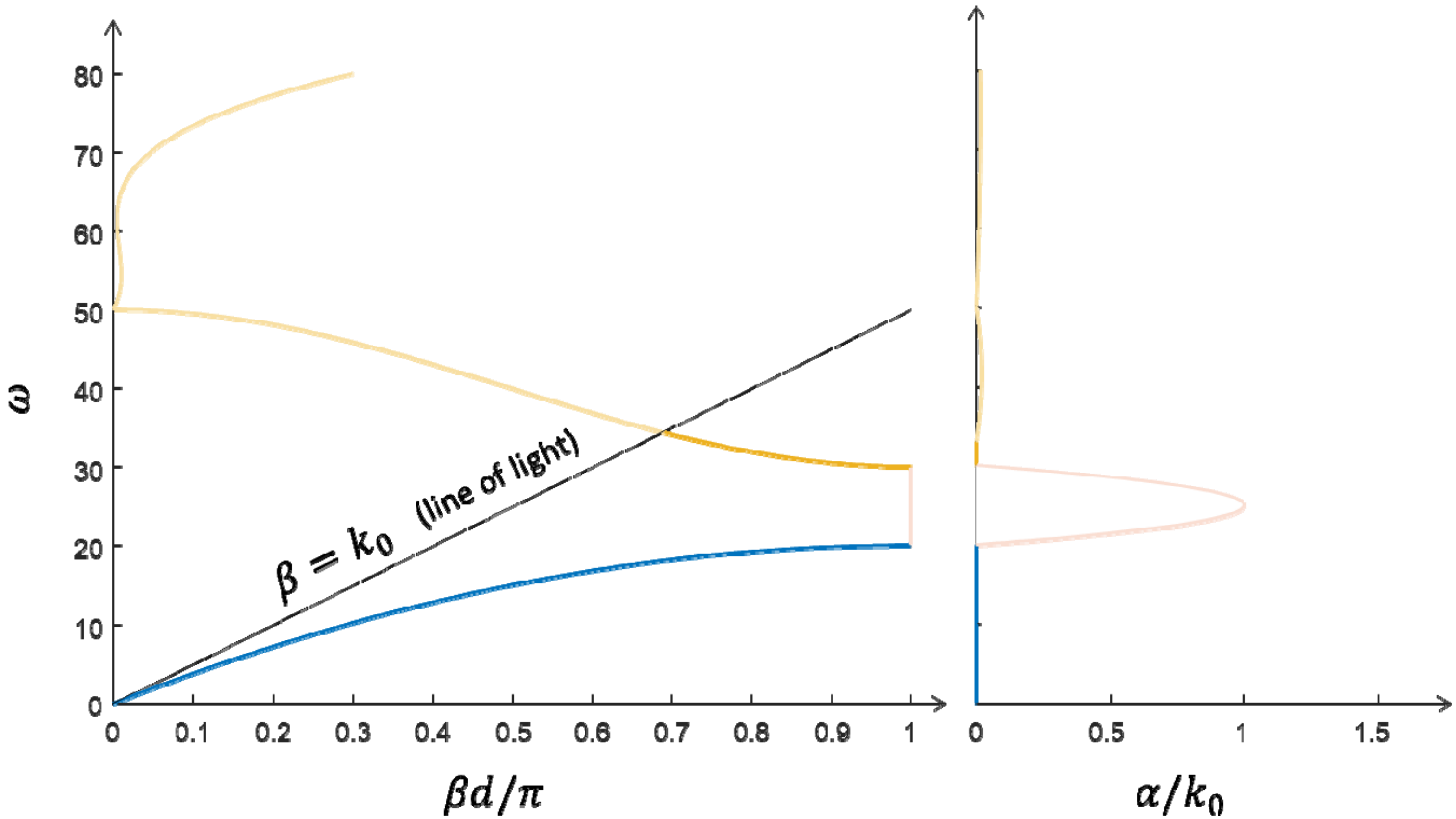
The wave propagates far from the structure.

A fast wave in an open structure is a **radiating wave** (or **leaky wave**).



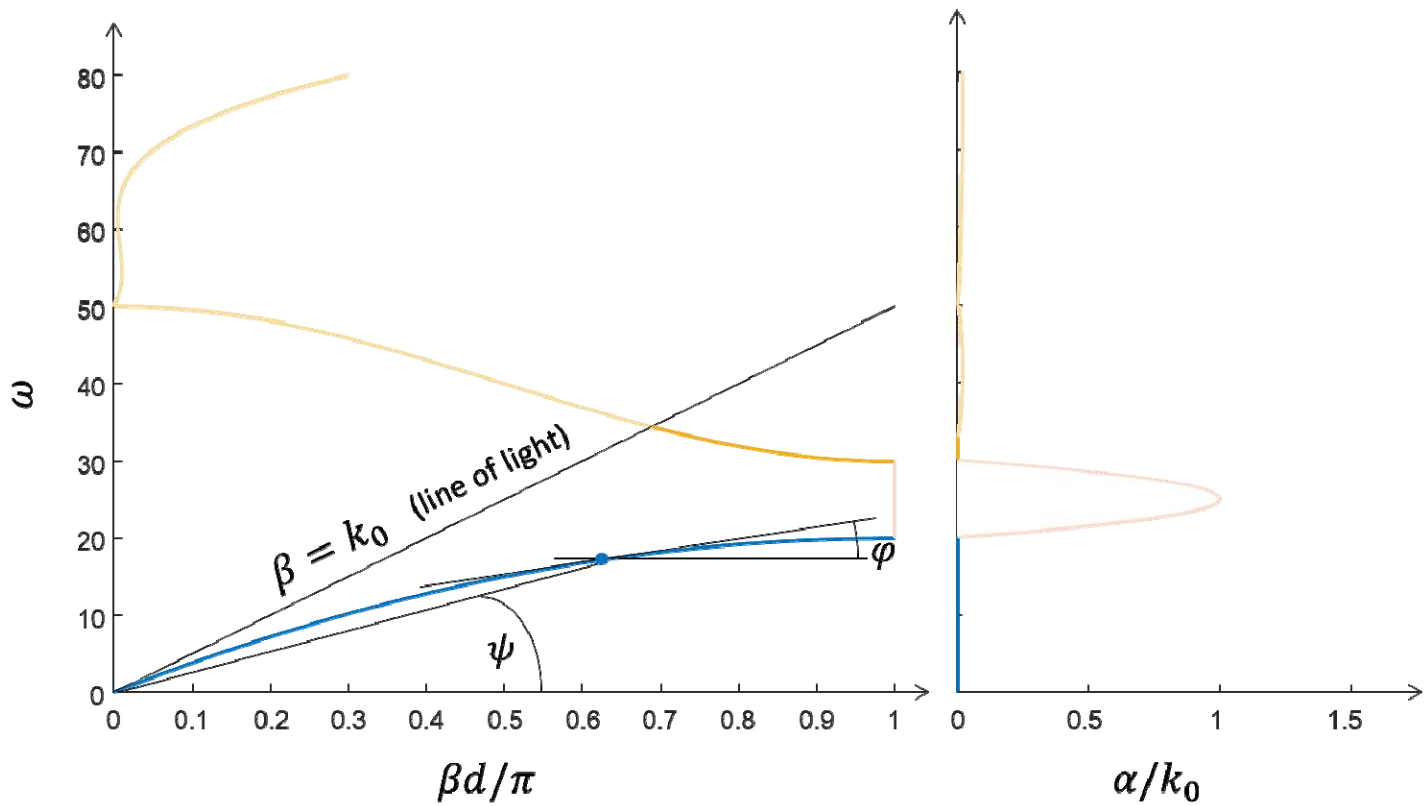
Propagating (or Real) Regimes 1/2

In a lossless structure, frequency bands where the attenuation constant $\alpha = 0$ allow for the propagation of waves without any attenuation.



If losses are present, an attenuation appears also in propagating regimes.

Propagating (or Real) Regimes 2/2

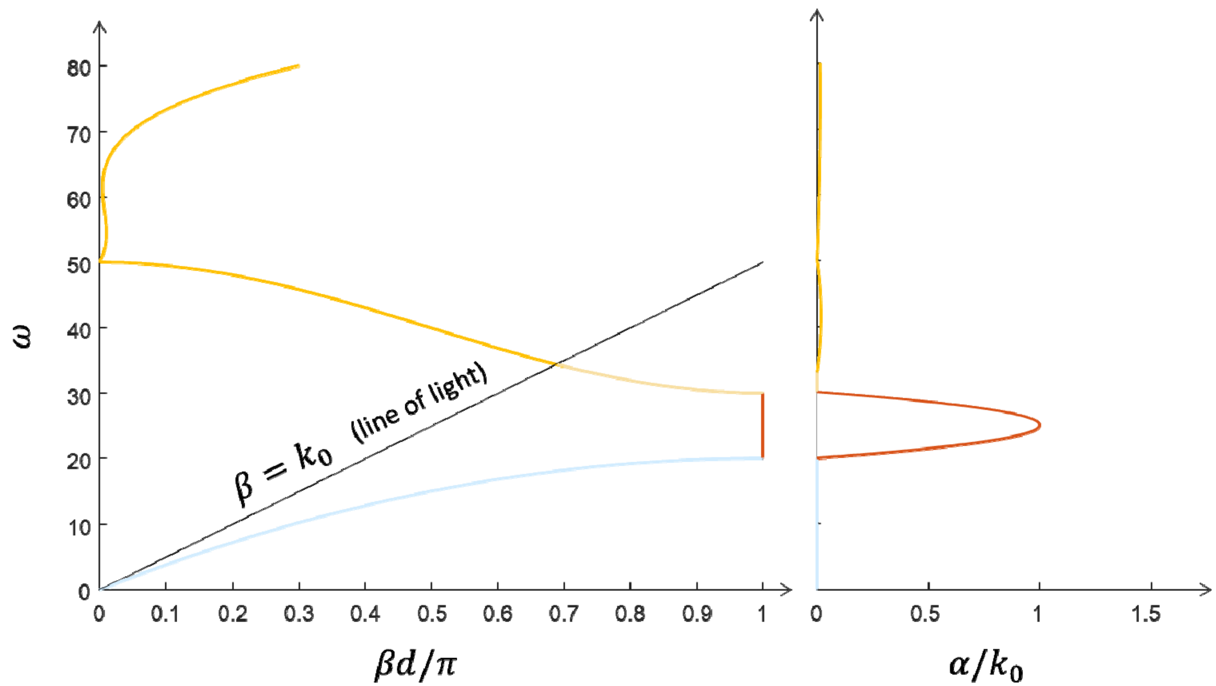


Phase velocity $v_p = \omega/\beta$ can be visualized through the angle ψ: every harmonic has its v_p
Group velocity $v_g = d\omega/d\beta$ can be visualized through the angle φ: all harmonics share the same v_g

A **lack of frequency dispersion** (constant $v_g = v_p$) corresponds to a **straight line**, i.e. a linear phase constant.

Attenuated (or Complex) Regimes

In a lossless structure, there are frequency bands where the phase constant is fixed with frequency and the attenuation constant a varies.



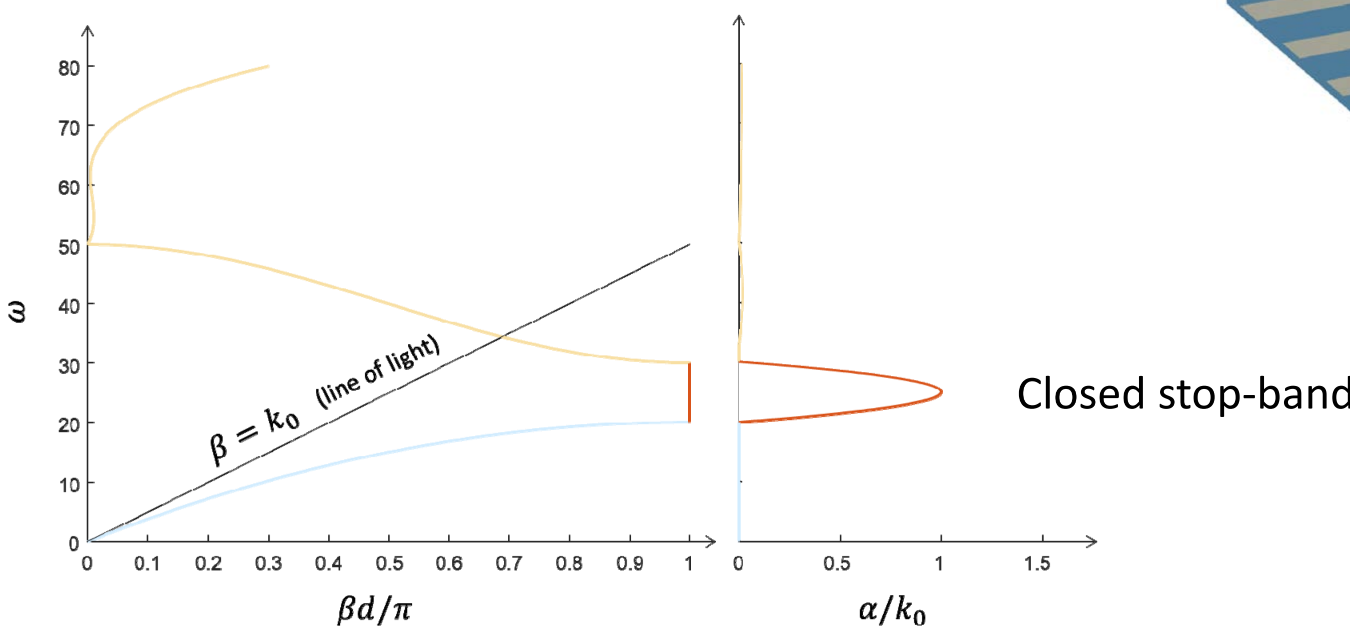
These are complex modes which do not carry real power: they are attenuating regimes.

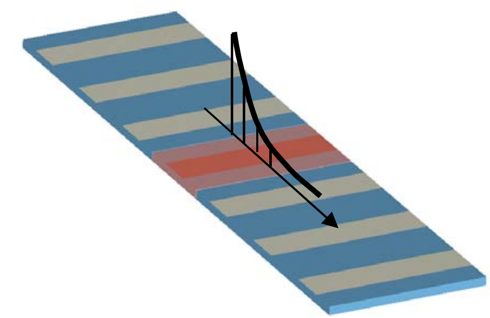
In commercial software, stopbands appear as gaps between propagating regimes.
However, the calculation of α can be useful to evaluate the attenuation across the structure.

The Closed Stop Band

The stop-band usually present at $\beta=\pi/d$ is also called "closed stop band":
A TEM mode lightly perturbed by periodic loads has its closed stop band in the bound region.

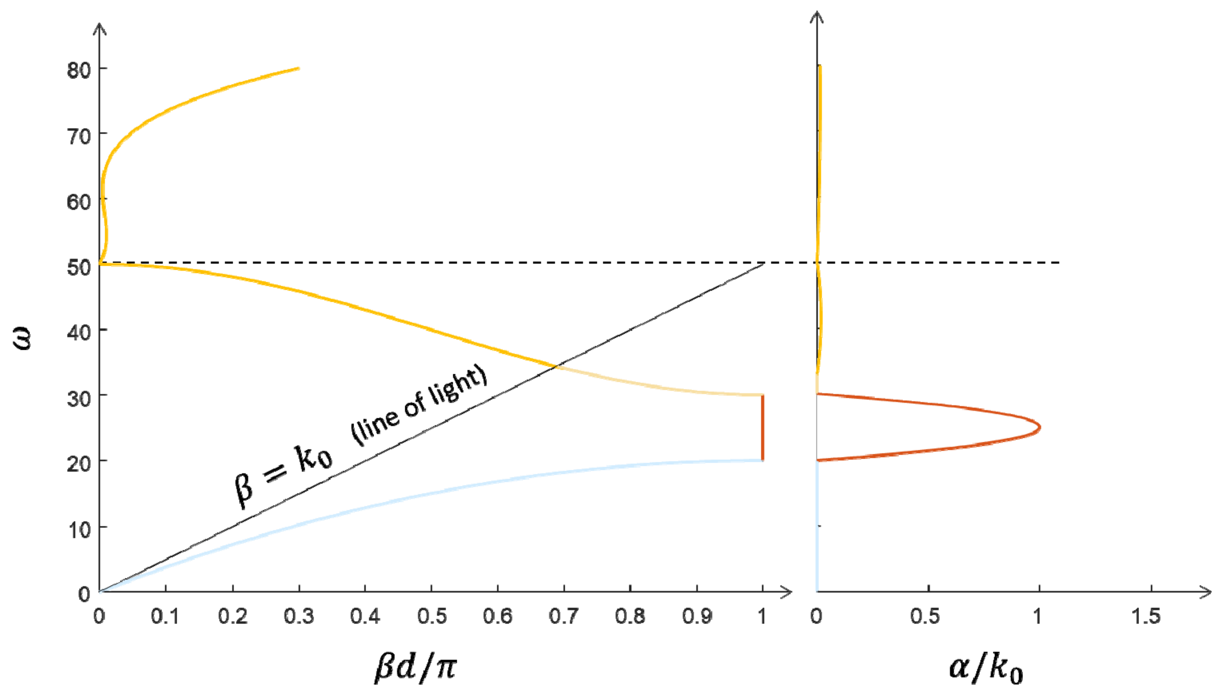
This frequency region is often considered as a "gap" between passbands.
It is used for its attenuation features as an electromagnetic band-gap (EBG) material.





The Open Stop Band

The stop-band usually present at $\beta = 0$ is also called "open stop band", because it lies in the radiating region when the structure is open.



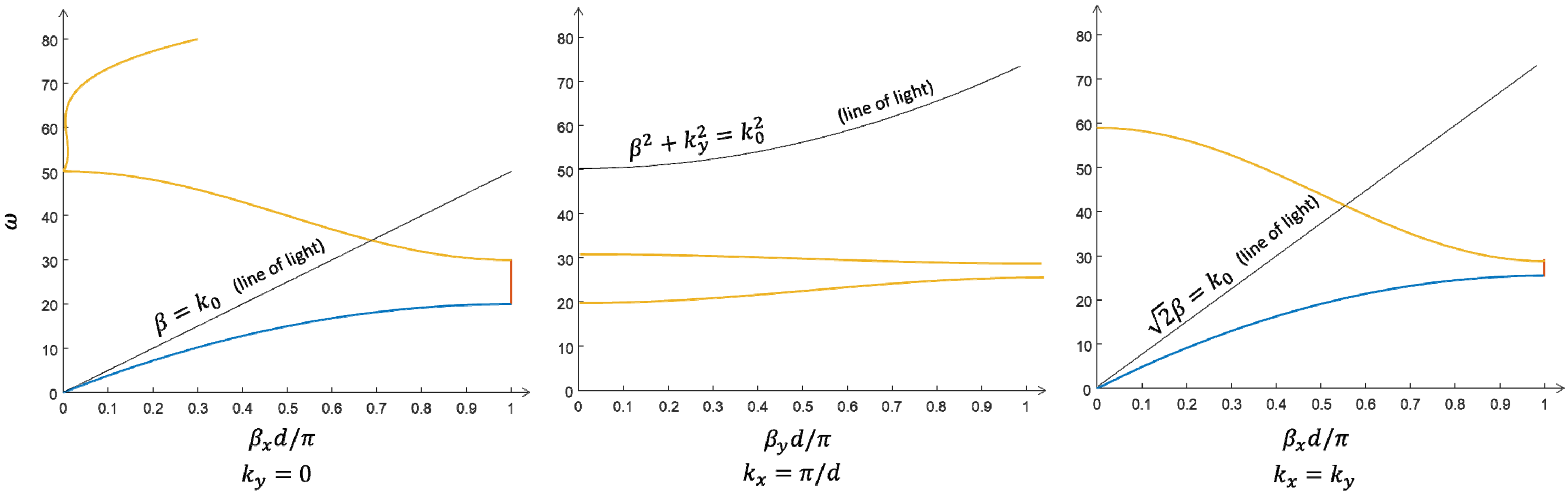
At the same frequency $\beta = 0$ and $\alpha = 0$. This phenomenon is responsible for a degradation of the radiation in broadside direction of a leaky wave antenna.

The Brillouin Diagram in 2-D 1/3

In 2-D periodic structures, the dispersive equation is of the kind

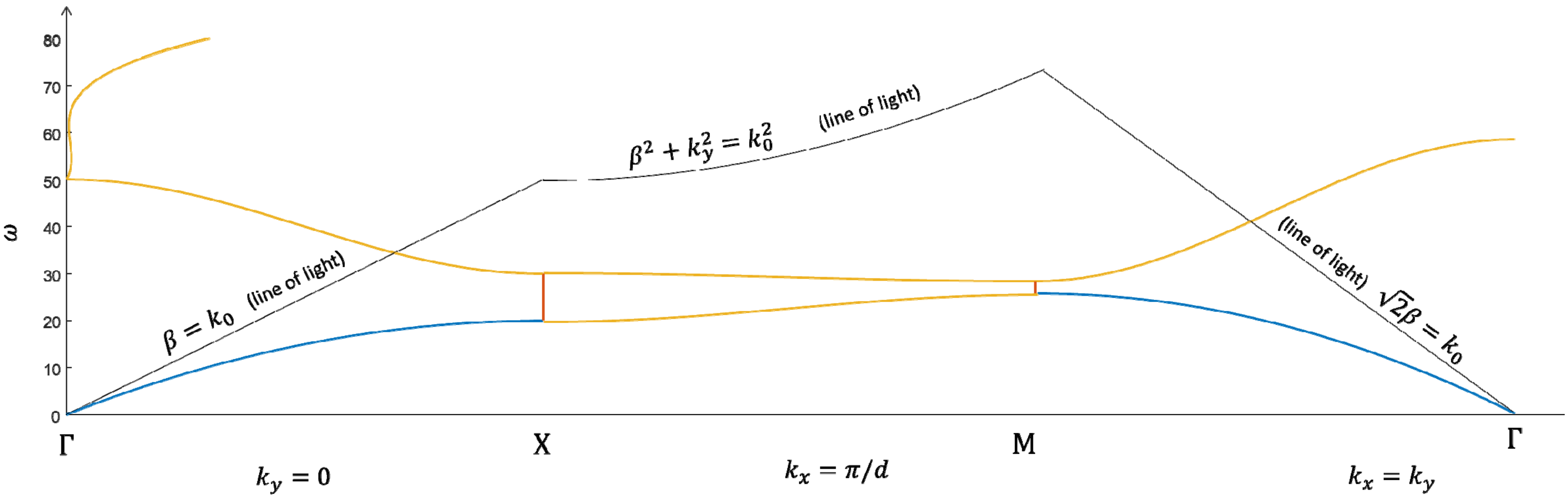
$$\omega = \omega(k_x, k_y)$$

The 2-D Brillouin diagram can be visualized by fixing k_y at different values

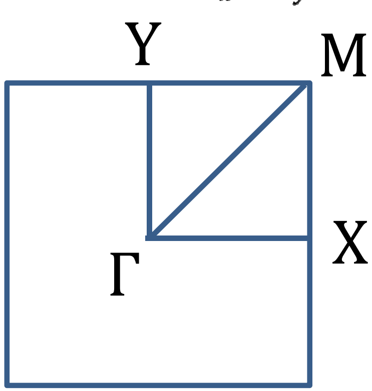


The Brillouin Diagram in 2-D 2/3

Putting all these slices on the same diagram:

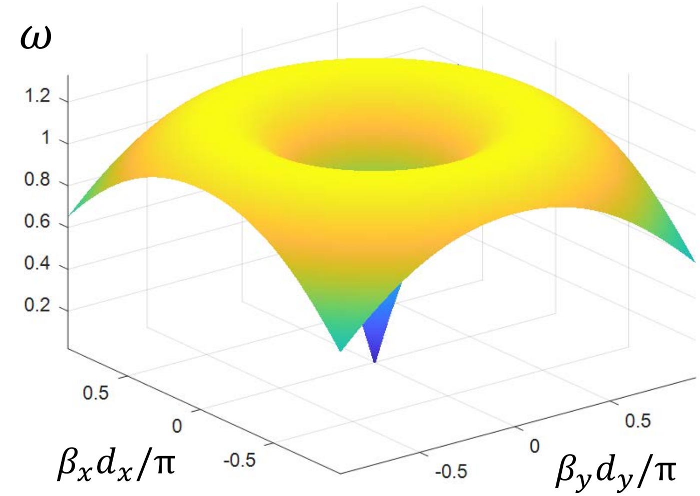


For rectangular lattices, the “slices” follow the boundary of an “irreducible Brillouin zone”: *the first Brillouin zone reduced by all of the symmetries in the point group of the lattice*.

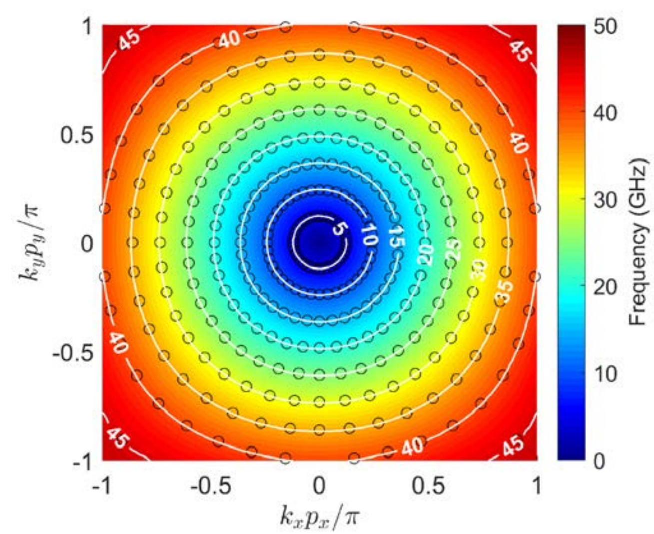


The Brillouin Diagram in 2-D 3/3

The relation $\omega = \omega(k_x, k_y)$, if restricted to real wavenumbers, can also define surfaces in the three-dimensional space $(\omega, \beta_x, \beta_y)$.



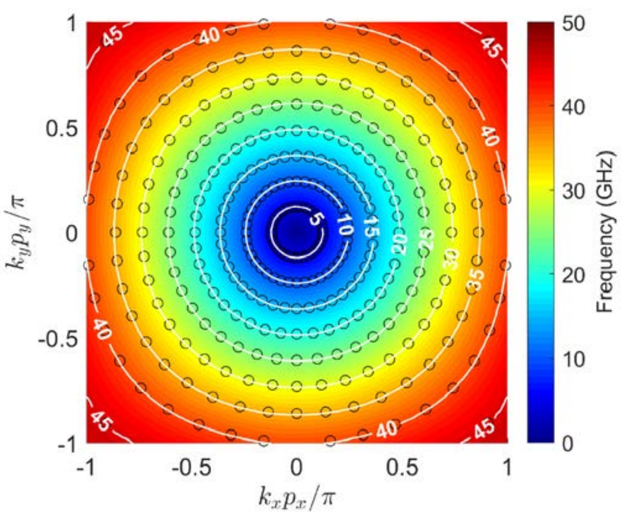
Iso-frequency curves show values (β_x, β_y) corresponding to a same frequency ω .



Anisotropy and Spatial Dispersion

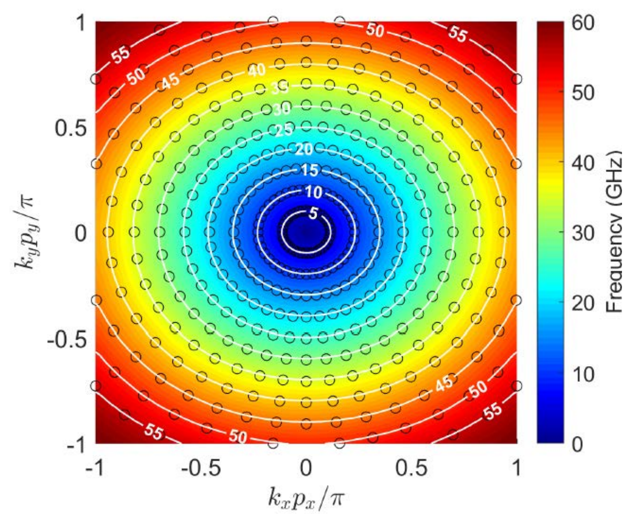
Iso-frequency curves are useful to determine if the periodic structure is equivalent to a material which is:

isotropic (circles)



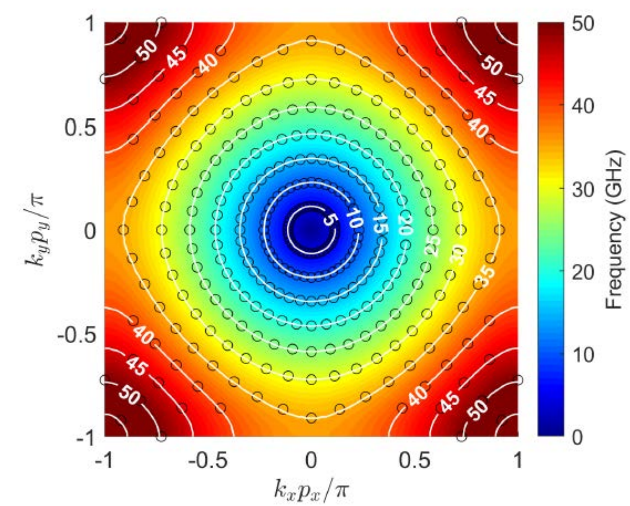
$$\omega^2 = a\beta_x^2 + a\beta_y^2$$

anisotropic (ellipses)



$$\omega^2 = a\beta_x^2 + b\beta_y^2$$

spatially dispersive



$$\omega = \omega(\beta_x, \beta_y)$$

O. Quevedo-Teruel, Q. Chen, F. Mesa, N. J. G. Fonseca and G. Valerio, "On the Benefits of Glide Symmetries for Microwave Devices," in *IEEE Journal of Microwaves*, vol. 1, no. 1, pp. 457-469, winter 2021.

Summary

- ❑ Definition of periodic structures in electromagnetics
- ❑ Eigenproblems and Bloch modes
- ❑ Formulation of a periodic problem: the periodic Green's function
- ❑ Solution of a periodic problem: the dispersion equation
- ❑ The Brillouin diagram in 1-D: bound and radiating modes, propagating and attenuating modes, closed and open stop band
- ❑ The Brillouin diagram in 2-D: the irreducible Brillouin zone, dispersion surfaces, and iso-frequency curves