Spatial Symmetries in Electromagnetics and Properties of Higher-Order Symmetries

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Simple Spatial Symmetries and Maxwell Equations
Very common geometrical symmetries in electromagnetics are defined with respect to a plane, here \((x, y)\).

Let a structure be invariant under the mirroring around the \(z = 0\) plane: \(M:\)

\[
\begin{align*}
x &\rightarrow x \\
y &\rightarrow y \\
z &\rightarrow -z
\end{align*}
\]
Symmetry with Respect to a Plane

The modal fields of the structures are solutions of the eigenproblem

\[ ME_i = \lambda E_i \]

Since \( M^2 \) is the identity operator, \( \lambda = \pm 1 \)

\[ ME_Z = E_Z \]
\[ E_Z(-z) = E_Z(z) \]

Even \( E_Z \) field

(\( z = 0 \) plane is an equivalent PEC)

\[ ME_Z = -E_Z \]
\[ E_Z(-z) = -E_Z(z) \]

Odd \( E_Z \) field

(\( z = 0 \) plane is an equivalent PMC)

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Symmetry with Respect to a Plane: PEC

\[ ME_z = E_z \quad \text{Even } E_z \text{ field} \]

If \( E_z \) is even, Maxwell equations are satisfied by

Even \( H_x, H_y \)  
Odd \( E_x, E_y, H_z \)

\[ MH_x = H_x, MH_y = H_y \]
\[ ME_x = -E_x, ME_y = -E_y, MH_z = -H_z \]

so \( E_x(z = 0) = 0, E_y(z = 0) = 0 \quad \rightarrow \quad z = 0 \text{ is a PEC plane} \)

If electric (magnetic) sources have the same parity as E (H) fields, only this parity is excited: the \( z = 0 \) plane is equivalent to a PEC.
Symmetry with Respect to a Plane: PMC

\[ ME_z = -E_z \quad \text{Odd } E_z \text{ field} \]

If \( E_z \) is odd, Maxwell equations are satisfied by

\begin{align*}
\text{Odd } H_x, H_y & \quad MH_x = -H_x, MH_y = -H_y \\
\text{Even } E_x, E_y, H_z & \quad ME_x = E_x, ME_y = E_y, MH_z = H_z
\end{align*}

so \( H_x(z = 0) = 0, H_y(z = 0) = 0 \) \( \Rightarrow \) \( z = 0 \) is a PMC plane

If electric (magnetic) sources have the same parity as \( E \) (\( H \)) fields, only this parity is excited: the \( z = 0 \) plane is equivalent to a PMC
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Higher Order Spatial Symmetries in Periodic Structures
A glide symmetric structure is the invariant under the composition of
- a **mirroring**
- a **shift of half a period** in a periodic structure

The symmetry plane \( z = 0 \) is called the glide plane.

Since two glide transformations are a translation of \( d_x \), a glide structure is also periodic.
Glide-Symmetry Along Two Directions

Conventional (Mirror-Symmetric)

Glide-Symmetric


2-D glide operator

\[
G: \begin{cases} 
  x \to x + d_x/2 \\
  y \to y + d_y/2 \\
  z \to -z
\end{cases}
\]
The glide operator $G$, not the translation operator $T$, is the minimal symmetry of the structure and determines its dispersion properties.

$$GE_i = \lambda E_i \quad \Rightarrow \quad TE_i = G^2 E_i = \lambda^2 E_i \quad \Rightarrow \quad \lambda = \pm e^{-jkx \frac{dx}{2}}$$
Brillouin Diagram of Non-Glide Symmetric Structures

\[ T E_i = e^{-j k_x d_x} E_i \]
\[ \kappa_{x,n} = \kappa_x + \frac{2\pi n}{d_x} \]

Different harmonics couple when meeting at the same frequency

Stop bands at \( \beta d = n\pi \), \( n \) odd and even
Brillouin Diagram of Glide Symmetric Structures

\[ G E_i = +e^{-j k_x \frac{d_x}{2}} E_i \]

\[ \kappa_{x,n}^{(1)} = k_x + \frac{4\pi n}{d_x} \]

Different harmonics (1) couple only at \( \beta d = n\pi , n = \pm 2, \pm 6, \ldots \)
Brillouin Diagram of Glide Symmetric Structures

\[ GE_i = +e^{-j k_x \frac{d_x}{2}} E_i \]

\[ GE_i = -e^{-j k_x \frac{d_x}{2}} E_i \]

\[ \beta d = n\pi , n = \pm 2, \pm 6, \pm 10, \ldots \]

\[ \beta d = n\pi , n = \pm 4, \pm 8, \pm 12, \ldots \]
Brillouin Diagram of Glide Symmetric Structures

\[ G E_i = +e^{-jkx \frac{dx}{2}} E_i \]

\[ k_{x,n}^{(1)} = k_x + \frac{4\pi n}{d_x} \]  \hspace{2cm} (1)

\[ G E_i = -e^{-jkx \frac{dx}{2}} E_i \]

\[ k_{x,n}^{(2)} = k_{x,n}^{(1)} + \frac{2\pi n}{d_x} \]  \hspace{2cm} (2)

Different harmonics (1) couple only at \( \beta d = n\pi \), \( n = \pm 2, \pm 6, \pm 10, \ldots \)

Different harmonics (2) couple only at \( \beta d = n\pi \), \( n = \pm 4, \pm 8, \pm 12, \ldots \)

Harmonics (1) and (2) do not couple since they have different z-symmetry! (to be seen later...)

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Brillouin Diagram of Glide Symmetric Structures

\[ GE_i = +e^{-jk_x \frac{d_x}{2} E_i} \quad \Rightarrow \quad k_{x,n}^{(1)} = k_x + \frac{4\pi n}{d_x} \]  

(1)

\[ GE_i = -e^{-jk_x \frac{d_x}{2} E_i} \quad \Rightarrow \quad k_{x,n}^{(2)} = k_{x,n}^{(1)} + \frac{2\pi n}{d_x} \]  

(2)

Glide-symmetric structures only have stop bands at \( \beta d = n\pi \), \( n \) even

No stop band is present at \( \beta d = \pi \) (closed stop-band, X point)
The absence of a stop-band at $\beta d = \pi$ (X point) is typical of glide-symmetric structures.

Analogous to the open-stop band suppression at $\beta = 0$ for broadside-radiating leaky waves.
Symmetry Properties of Harmonics 1/2

\[ GE_i = +e^{-jk_x \frac{d_x}{2}} E_i \quad \rightarrow \quad k_{x,n}^{(1)} = k_x + \frac{4\pi n}{d_x} \quad (1) \]

\[ GE_i = -e^{-jk_x \frac{d_x}{2}} E_i \quad \rightarrow \quad k_{x,n}^{(2)} = k_{x,n}^{(1)} + \frac{2\pi n}{d_x} \quad (2) \]

We can associate

- solutions (1) to harmonics having even integer \( n \). Note that \( e^{-jk_{x,n} \frac{d_x}{2}} = +e^{-jk_x \frac{d_x}{2}} \)
- solutions (2) to harmonics having odd integer \( n \). Note that \( e^{-jk_{x,n} \frac{d_x}{2}} = -e^{-jk_x \frac{d_x}{2}} \)

So we can state \( e^{-jk_{x,n} \frac{d_x}{2}} = (-1)^n e^{-jk_x \frac{d_x}{2}} \)
Let us apply the glide operator to a Bloch mode:

\[ GE_i = G \sum_{n=-\infty}^{+\infty} c_n(z) e^{-j k_x n x} = \sum_{n=-\infty}^{+\infty} c_n(-z) e^{-j k_x n (x + \frac{d_x}{2})} = \sum_{n=-\infty}^{+\infty} c_n(-z) e^{-j k_x n x} e^{-j k_x \frac{d_x}{2}} \]

\[ = \sum_{n=-\infty}^{+\infty} c_n(-z) e^{-j k_x x} e^{-j k_x \frac{d_x}{2}} = \sum_{n=-\infty}^{+\infty} c_n(-z) e^{-j k_x n x} (-1)^n e^{-j k_x \frac{d_x}{2}} \]

\[ = e^{-j k_x \frac{d_x}{2}} \sum_{n=-\infty}^{+\infty} (-1)^n c_n(-z) e^{-j k_x n x} = e^{-j k_x \frac{d_x}{2}} E_i \]

only if \( c_n(z) = (-1)^n c_n(-z) \)

**Result 1:** Harmonics with different \( n \)-parity have different \( z \)-parity and do not couple.

**Result 2:** The glide plane is equivalent to a PEC or PMC according to the \( n \)-parity of harmonics.
Glide Symmetry vs. Non-Glide Symmetry: Dispersion

Glide structures are less dispersive in frequency than non-glide ones!
Application: Luneburg Lens with Glide-Symmetric Holes

Glide Symmetry vs. Non-Glide Symmetry: Stopband

Glide structures have a wider stopband than non-glide ones!
This technology can be employed to design a number of electromagnetic components:

- Flanges.
- Waveguides.

Twist-Symmetric Periodic Structures

Eigenvalue problem for twist symmetries:

\[ S_m[E(\rho, \varphi, z)] = E\left(\rho + \frac{p}{m}, \varphi + \frac{2\pi}{m}, z\right) = e^{-jk_x \frac{p}{m}} E(\rho, \varphi, z) \]

\( k_x p \) is periodic of \( 2m\pi \)
Twist Symmetry: Dispersion Diagrams

Single hole

3-fold twist

$k_T p / \pi$

$k_{S_3} p / \pi$

\[ d = 1.5 \text{ mm}, \quad \ell = 2.4 \text{ mm}, \quad p = 12 \text{ mm}, \quad g = 0.1 \text{ mm} \]
Polar-Glide Symmetry

- Coaxial cable with rings inside and outside metallic conductors.
- Similar approach of transformation optics.

Polar glide-symmetric structure remains invariant under a translation and a reflection to a circular or cylindrical surface

\[ S_{PG} \equiv \begin{cases} 
\rho \rightarrow \frac{R^2}{\rho} \\
\theta \rightarrow \theta + \alpha \\
z \rightarrow z + \frac{p}{2} 
\end{cases} \]

Polar-Glide Symmetry

- Control of the stop-band in the propagation of the coaxial cable.

- Elimination of the first band-gap with polar glide.

Summary

- Review of Maxwell equations under planar symmetries.
- Definition of glide (1-D and 2-D) and twist symmetries.
- Dispersion properties of higher order symmetries in terms of harmonic coupling
- Properties of higher-order symmetric structures: dispersion and stopbands
- Polar-glide symmetries