Symmetry in electromagnetism: group theory and Maxwell's equations

Simon Horsley

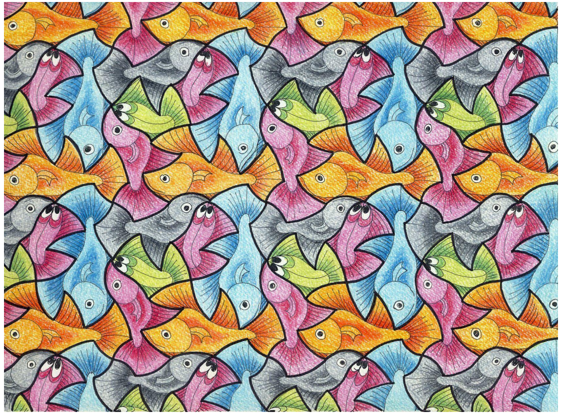
UNIVERSITY OF EXETER

Part 1: A sketch of group theory

Symmetry: nature, art, and physics

Rotational symmetry

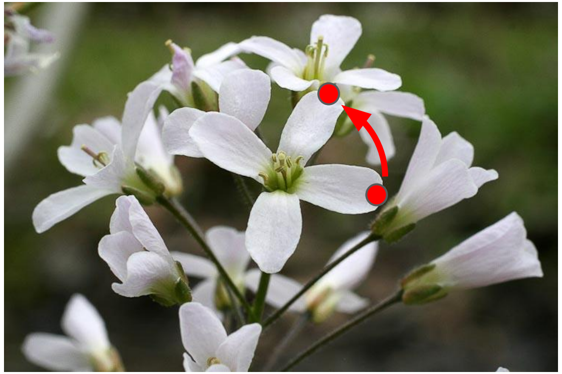
Reflection symmetry



Translational symmetry

Permutation symmetry

Symmetry: nature, art, and physics



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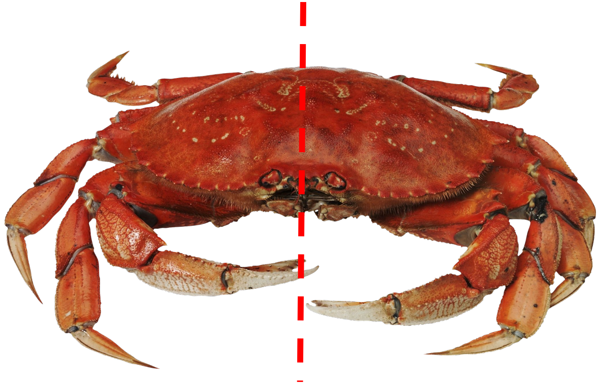
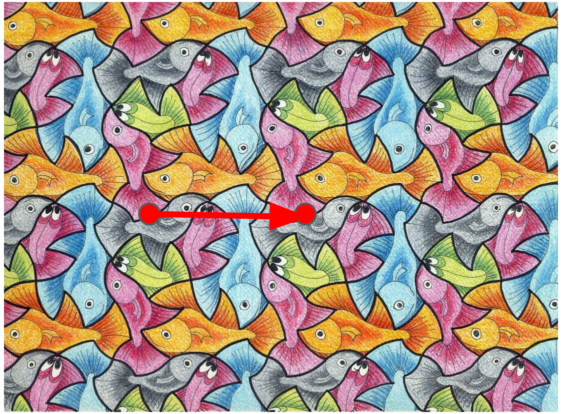


Reflection symmetry



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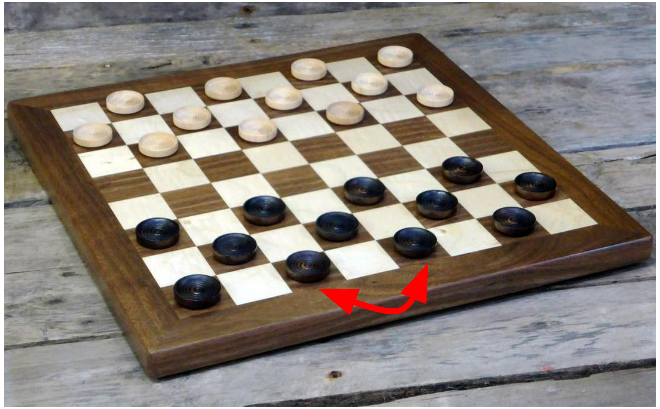


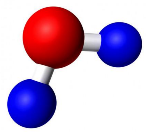
Permutation symmetry

Symmetry: nature, art, and physics

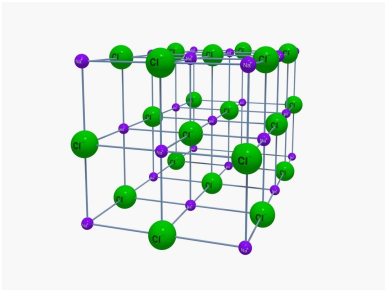
Symmetry in physics:

- Discrete rotation and translation symmetry of atoms molecules and crystals. (Quantum chemistry)
- Time and space inversion symmetry (e.g. Newton's equations).
- Electric/magnetic duality symmetry (two polarization of EM waves).
- Continuous translation and rotation symmetry (Conservation laws).
- Local continuous gauge symmetry and diffeomorphism invariance (Standard Model and General Relativity)

Symmetries of matter



H_2O



NaCl

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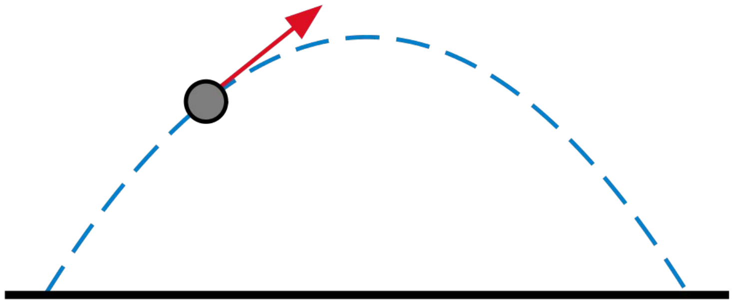
Time symmetric motion



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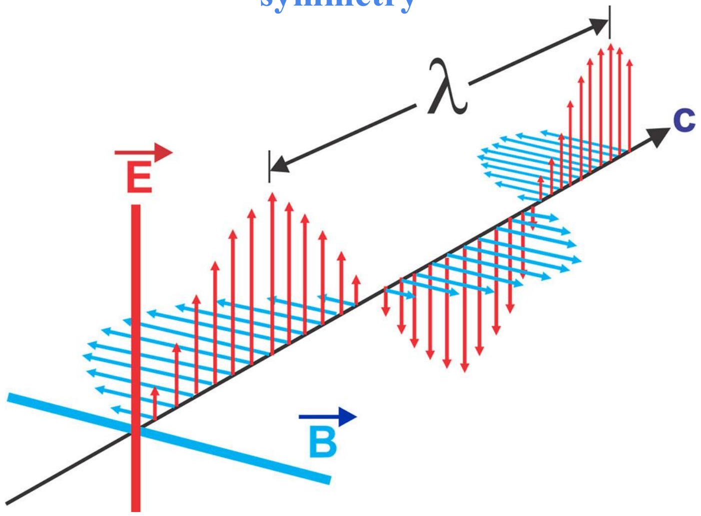
Electric-magnetic symmetry



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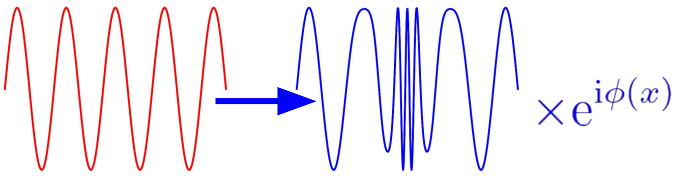
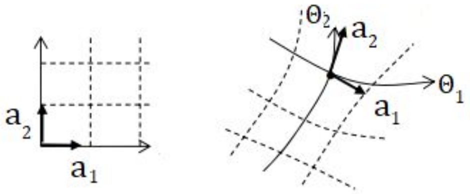
Gauge theory and general covariance



A_x $\qquad$ $A_x - \frac{e}{\hbar}\partial_x \phi$



Cartesian $\qquad$ **Curvilinear**

$$R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu = \frac{8\pi G}{c^4} T^\mu_\nu$$

How to describe symmetry

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Image as a vector: $|\psi\rangle = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \dots \\ p_N \end{pmatrix}$

Split image up into N pixels

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Split image up into N pixels

A symmetry *group* is the collection of *all* the matrices M that leave the object unchanged.

Transformed image: $|\psi'\rangle = \hat{M}|\psi\rangle = \begin{pmatrix} p'_0 \\ p'_1 \\ p'_2 \\ \cdots \\ p'_N \end{pmatrix}$

Symmetry if: $|\psi'\rangle = |\psi\rangle$

How to describe symmetry

Simpler example:

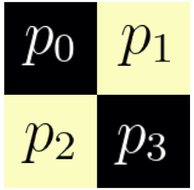

Image as a vector: $|\psi\rangle = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \dots \\ p_N \end{pmatrix}$

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Group theory

What is a `group'?

1. *Associative multiplication* $A(BC) = (AB)C$

2. *Closure (you can't get something not in the group from combining elements)*

3. *Inverse (for every* A *there is an* A^{-1} *such that* $A^{-1}A = E$*)*

4. *Identity (the group contains the identity* E*)*

A group is a
'bag' of matrices

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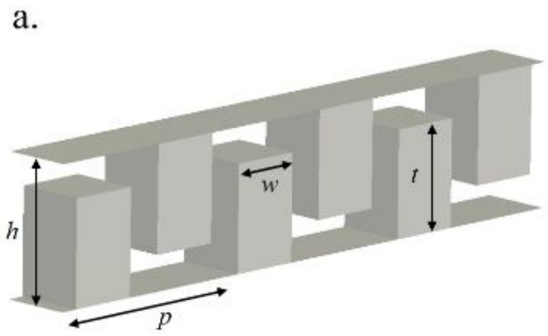
Proof that we have a group in our example:

Group theory

Example - higher symmetry waveguide:



O. Quevedo-Teruel, G. Valerio, Z. Sipus, and E. Rajo-Iglesias, "*Periodic structures with higher symmetries and their applications in electromagnetic devices*" IEEE Microwave Magazine (2020)

Symmetry group elements:

Group theory

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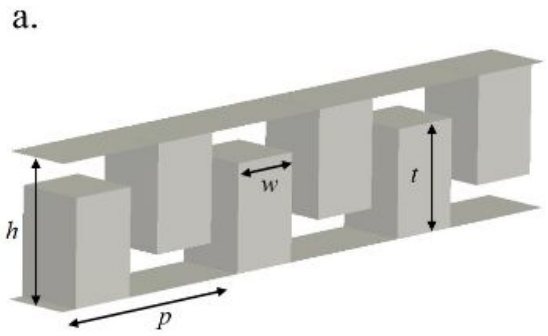

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Symmetry group elements:

This is a so-called '*Frieze group*', denoted *p11g*. See e.g. Conway et. al. "*The symmetries of things*"

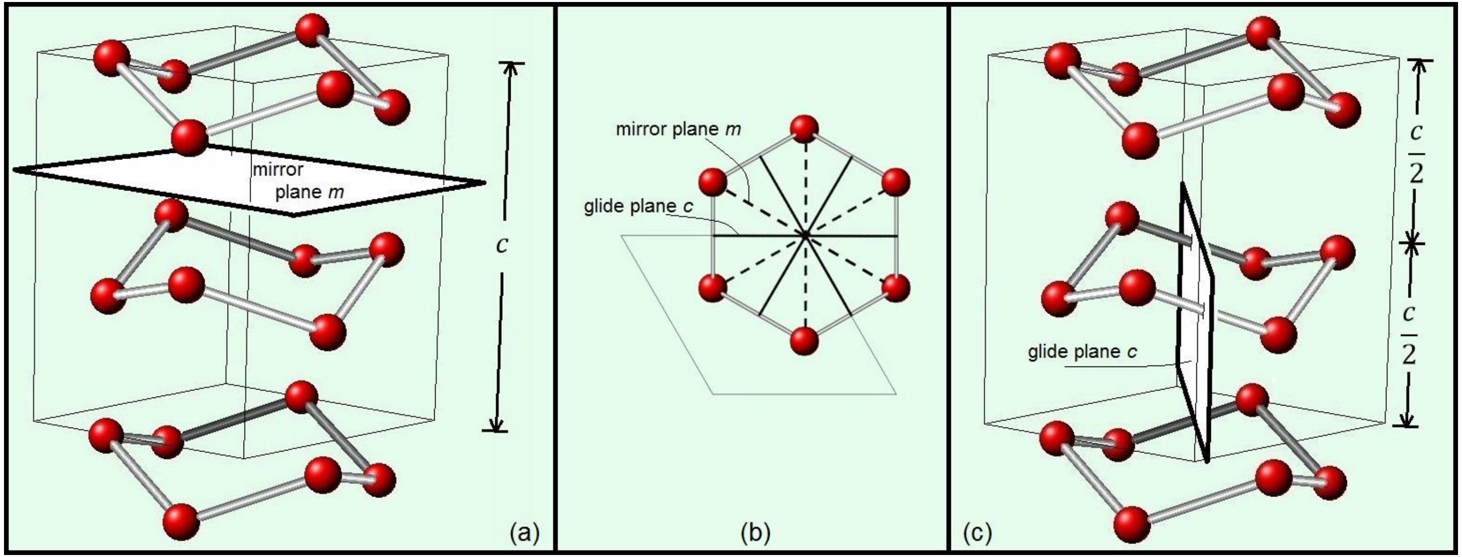
Group theory

More complicated glide symmetry:

"Hexagonal Ice" H_2O

Group theory

More complicated glide symmetry: "Hexagonal Ice" H_2O



Has combinations of rotation, mirror plane, and translation symmetry operations. Characterized by one (called $P6_3/mmm$) of the 230 possible "*space groups*".

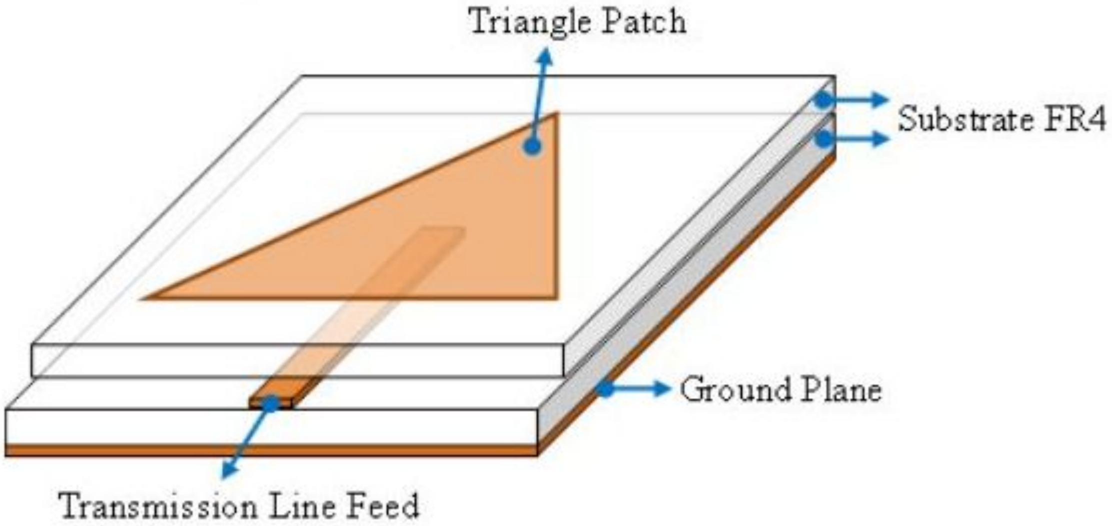
(See e.g. http://img.chem.ucl.ac.uk/sgp/)

Representations of groups

There are many '*isomorphic*' collections of symmetry group matrices. The same group structure can be realised in many different ways.

Example: C3 group (3 fold rotation)

C3	**M**	**M^2**	**E**
M	M^2	E	M
M^2	E	M	M^2
E	M	M^2	E



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Representation 2: $M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

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Representation 3: $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

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M^2	E	M	M^2
E	M	M^2	E

Representation 1: $M = \mathrm{e}^{2\pi \mathrm{i}/3}$

Representation 2: $$M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Representation 3: $$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

...

Irreducible representations

Q: How do we deal with this ambiguity of how to represent the group?

A: Find the 'irreducible representations'.

A representation is fully reducible if, for every matrix in the group we can perform a transformation $M' = TMT^{-1}$ such that,

$$M = \begin{pmatrix} \underbrace{\boldsymbol{M}_1}_{n_1 \times n_1 \text{ matrix}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \underbrace{\boldsymbol{M}_2}_{n_2 \times n_2 \text{ matrix}} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \mathbf{0} \\ \dots & \dots & \dots & \boldsymbol{M}_n \end{pmatrix}$$

The $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, …, $\boldsymbol{M}_n$ are each elements of an irreducible representation of the group. They are the "smallest" matrices that capture the group structure (the symmetry).

The dimensions of irreducible representations

A useful theorem: $\sum_\mu n_\mu^2 = g$

See e.g. M. Hamermesh “Group Theory and it’s application to Physical Problems” (Dover)

The dimensions of irreducible representations

A useful theorem: $\sum_{\mu} n_{\mu}^2 = g$

Example: find irreducible representations of C_3 rotation group:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Irreducible representations and degeneracy

The solutions to Maxwell's equations transform as irreducible representations of the relevant symmetry group

Symmetry operator generates a new solution to Maxwell's equations:

$$\hat{M}\begin{pmatrix} \boldsymbol{E}_n \\ c\boldsymbol{B}_n \end{pmatrix} = \sum_m M_{nm} \begin{pmatrix} \boldsymbol{E}_m \\ c\boldsymbol{B}_m \end{pmatrix}$$

Symmetry operator (e.g. glide transformation).

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Solution to Maxwell's equations, for e.g. fixed frequency ω.
One or more degenerate solutions.

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Transformed solution is also a solution to Maxwell's equations, for e.g. fixed frequency ω. But it is a different mixture of the degenerate solutions.

Irreducible representations and degeneracy

The solutions to Maxwell's equations transform as irreducible representations of the relevant symmetry group

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$$\hat{M}\begin{pmatrix}\boldsymbol{E}_n\\ c\boldsymbol{B}_n\end{pmatrix} = \sum_m M_{nm}\begin{pmatrix}\boldsymbol{E}_m\\ c\boldsymbol{B}_m\end{pmatrix}$$

$$\longrightarrow \quad \int d^3\boldsymbol{x}\,(\boldsymbol{E}_m^\star, c\boldsymbol{B}_m^\star)\,\hat{M}\begin{pmatrix}\boldsymbol{E}_n\\ c\boldsymbol{B}_n\end{pmatrix} = M_{nm}$$

The matrices formed from integrals between the symmetry operators and the solutions to Maxwell's equations form an irreducible representation of the symmetry group.

Example: degeneracy in a triangular waveguide

Maxwell's equations, simplified for the two polarizations:

***E* polarization:**

$$\nabla^2 E_z + (k_0^2 - k_z^2) E_z = 0$$

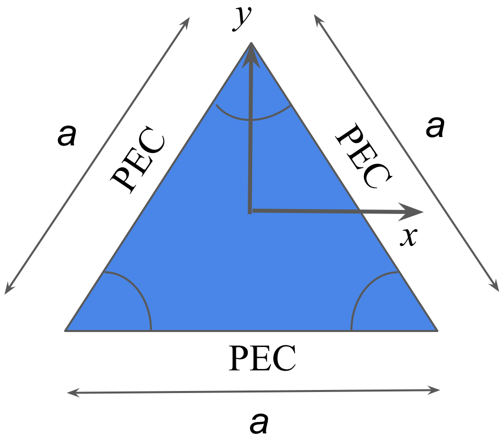
$E_z = 0$ (on PEC boundary)

***H* polarization:**

$$\nabla^2 H_z + (k_0^2 - k_z^2) H_z = 0$$

$\partial H_z / \partial n = 0$ (on PEC boundary)

After a bit of work we can solve this analytically...



Symmetry group (C3v, three-fold rotations):

Example: degeneracy in a triangular waveguide

Q: Without solving the equation, what is the degeneracy of the waveguide modes?

Find the dimension of the irreducible representations of the C3v group:

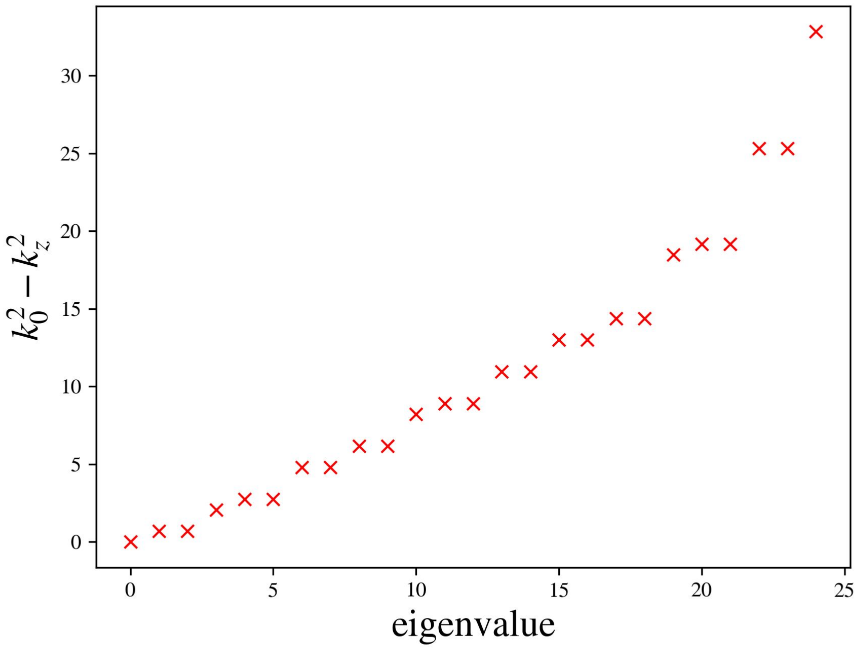
Our useful theorem: $$\sum_{\mu} n_{\mu}^2 = g$$

Example: degeneracy in a triangular waveguide

Exact solutions (quite a bit more work!)
see e.g. B. J. McCartin *SIAM Rev.* 45, 267 (2003)

Exact dispersion relation in triangular waveguide:

$$k_0^2 - k_z^2 = \frac{4}{27}\left(\frac{\pi}{r}\right)^2 [m^2 + n^2 + mn]$$



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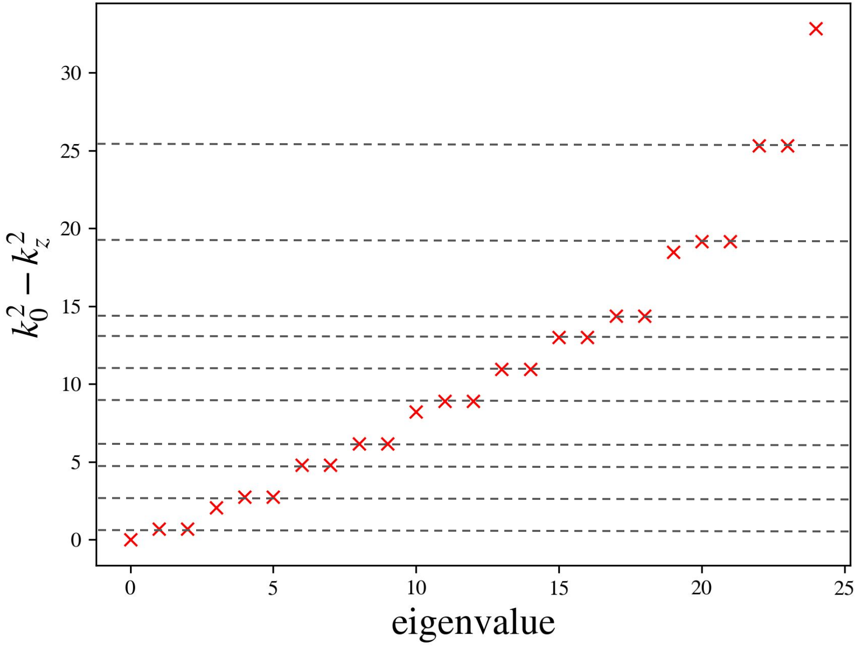
Eigenfunctions of 2 dimensional irreducible representation



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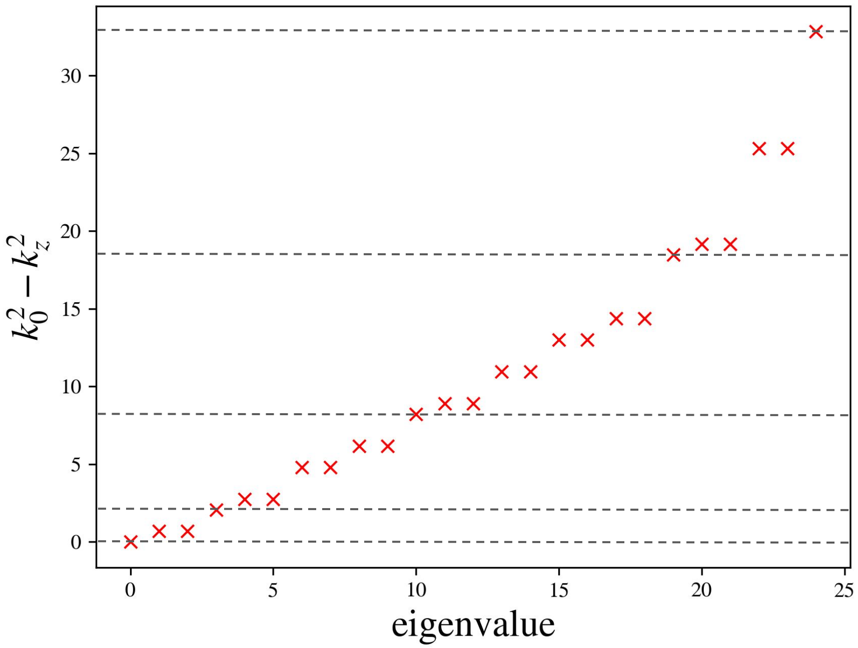
Eigenfunctions of 1 dimensional irreducible representation



5 minute break!

The story so far...

- Symmetry is an important principle in physics, and can be understood in terms of ***symmetry groups***.
- ***A group can be though of as a collection of matrices***, with each matrix corresponding to a transformation of the system (e.g. rotation of coordinates, translation, change of polarization…).
- A group satisfies ***four properties***; (1) ***associativity of multiplication***; (2) ***closure***; (3) every element has an ***inverse*** in the group; (4) the ***identity*** matrix is in the group.
- There are many different matrix ***representations*** of the same group structure.
- The ***irreducible representations*** are the "smallest" matrix representation of the group, and obey,

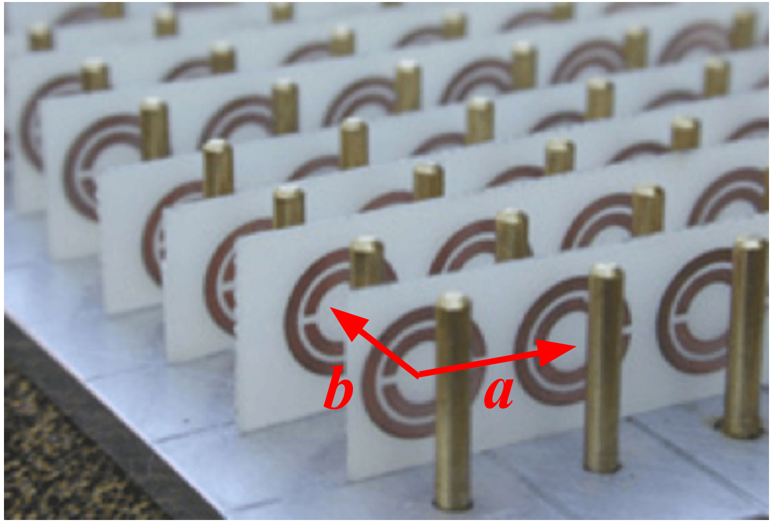
$$\sum_{\mu} n_{\mu}^{2} = g$$

- The ***dimensions of*** the matrices forming the ***irreducible representations*** tell us the possible ***degeneracies*** of e.g. the frequencies in the system.

Part 2: Examples of symmetry and group theory in electromagnetic materials

Translational symmetry and Bloch's theorem

Translational symmetry and Bloch's theorem

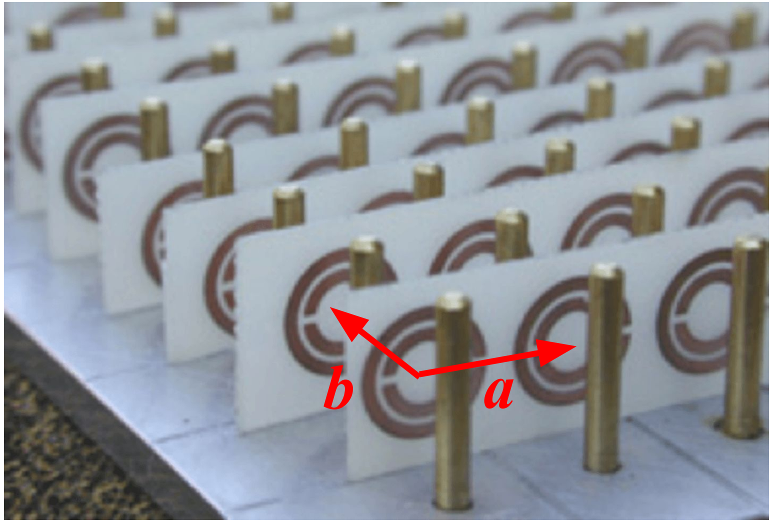


Symmetry transformation of lattice:

$$T_a : \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$$

$$T_b : \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{b}$$

Translational symmetry and Bloch's theorem



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$$(T_{\boldsymbol{b}})^n, (T_{\boldsymbol{a}})^n$$

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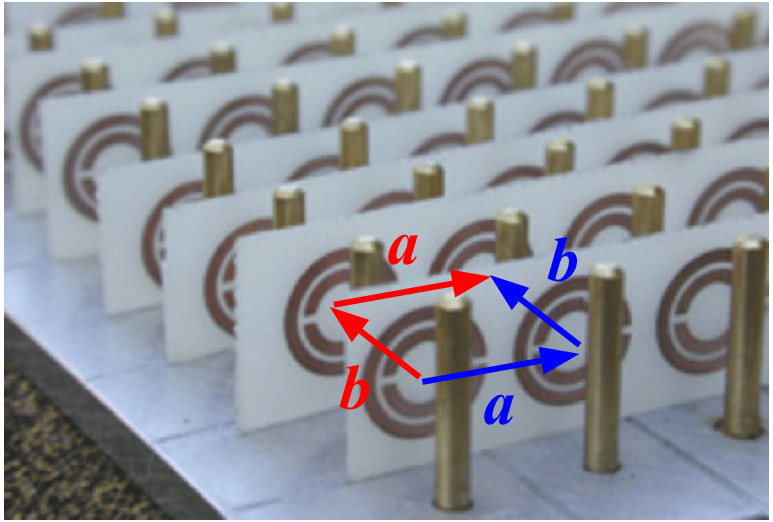

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$$[T_{\boldsymbol{a}}, T_{\boldsymbol{b}}] = T_{\boldsymbol{a}}T_{\boldsymbol{b}} - T_{\boldsymbol{b}}T_{\boldsymbol{a}} = 0$$

$\longrightarrow$ Matrices can all be simultaneously diagonalized

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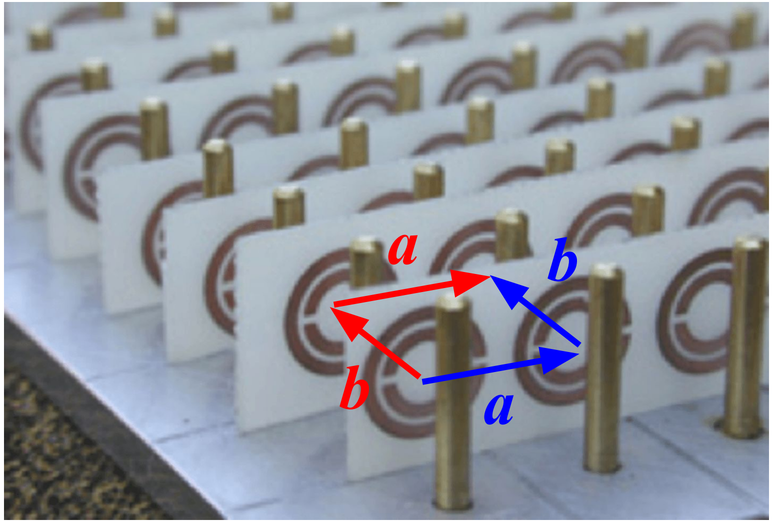

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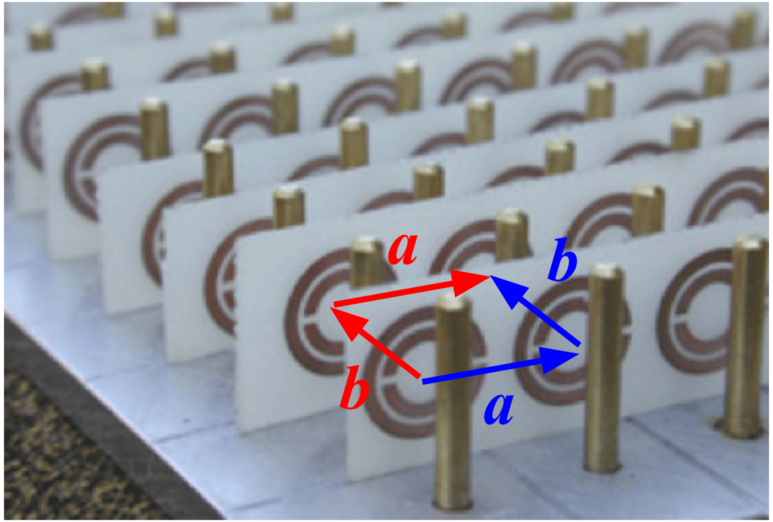

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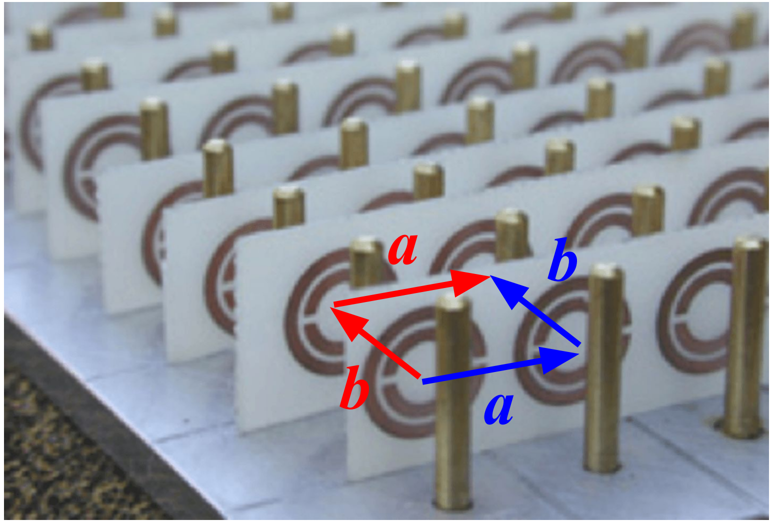
→ Matrices can all be simultaneously diagonalized

→ Irreducible representations are 1x1

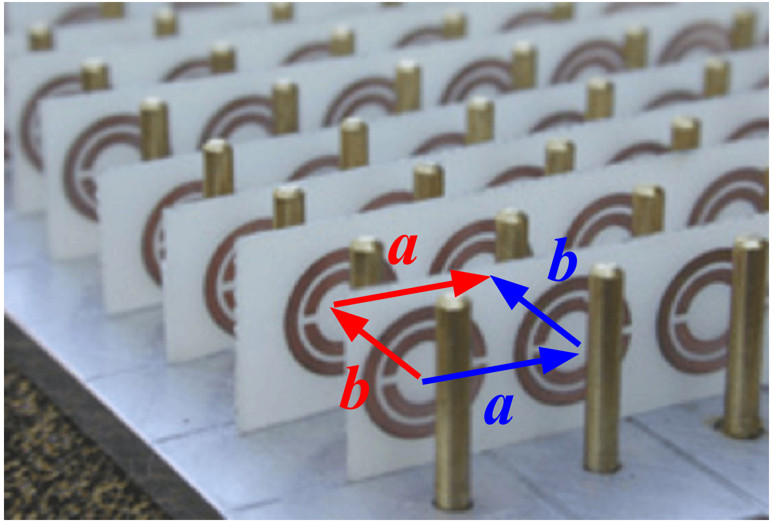
→ Symmetry transformation is just $E'_z = cE_z$

Translational symmetry and Bloch's theorem



Suppose the wave is the same after N translations along $\boldsymbol{a}$, and M translations along $\boldsymbol{b}$: i.e. $T_{\mathbf{a}}^{N}=1$ and $T_{\mathbf{b}}^{M}=1$

Translational symmetry and Bloch's theorem



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$$\longrightarrow \quad T_{\boldsymbol{a}} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}} = \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{a}}$$

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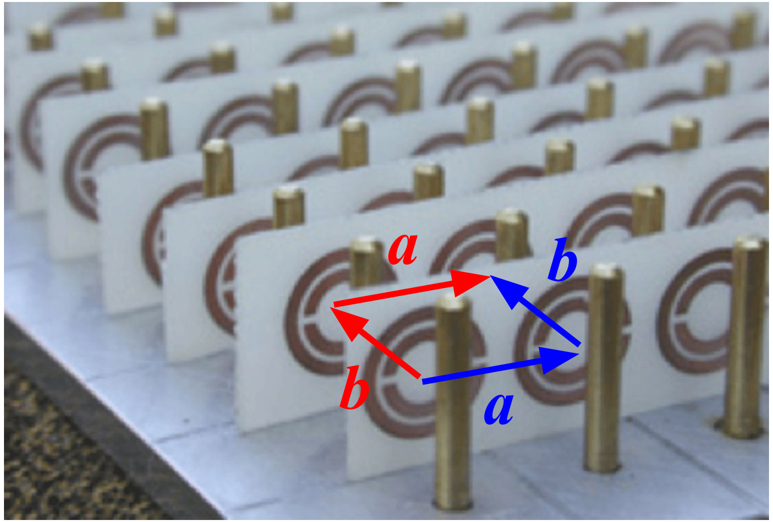
Translational symmetry and Bloch's theorem



Suppose the wave is the same after N translations along $\boldsymbol{a}$, and M translations along $\boldsymbol{b}$: i.e. $T_{\mathbf{a}}^{\mathrm{N}}=1$ and $T_{\mathbf{b}}^{\mathrm{M}}=1$

$$\longrightarrow \quad T_{\boldsymbol{a}} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{M}} = \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{a}}$$

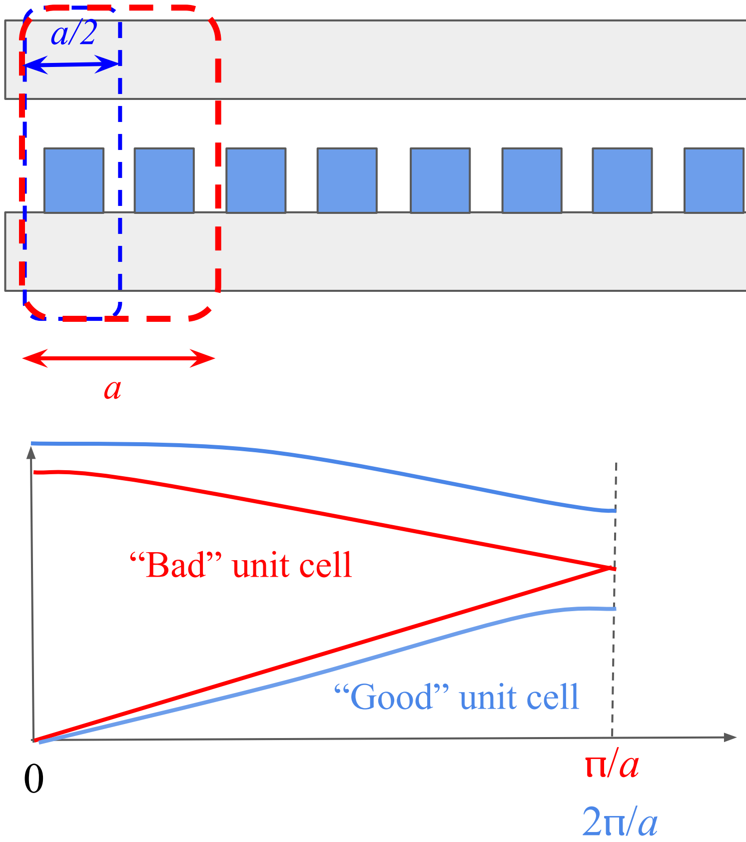
$$\longrightarrow \quad T_{\boldsymbol{b}} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{M}} = \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{b}}$$

This means that the electric field can be written as:

$$\boxed{E_z(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{x}} u_{\boldsymbol{K}}(\boldsymbol{x})}$$

Bloch's theorem from group theory!

Degeneracy in a badly chosen unit cell



At the Brillouin zone boundary the wave satisfies,

$$E_z(x+a) = -E_z(x)$$

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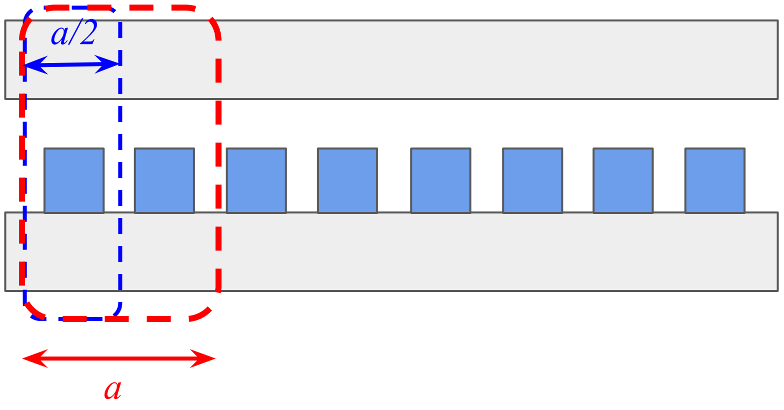

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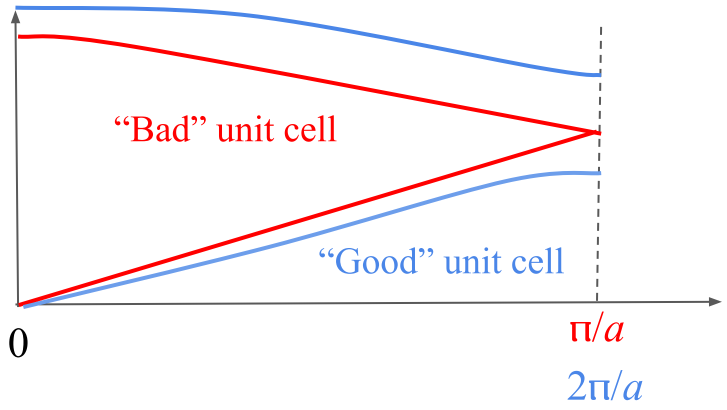
$$T_{a/2}^2 E_z = -E_z$$



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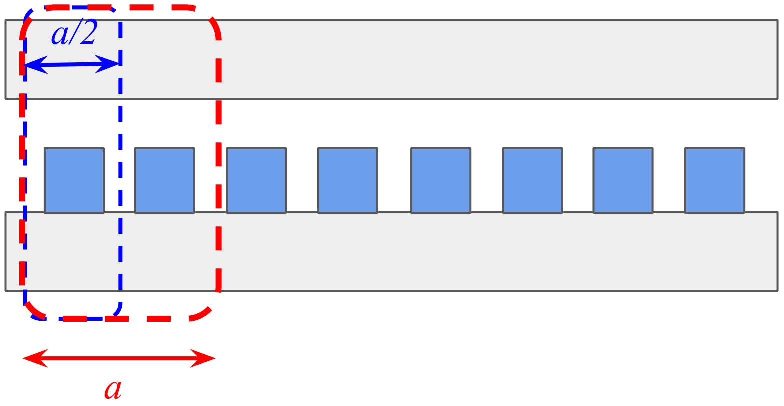

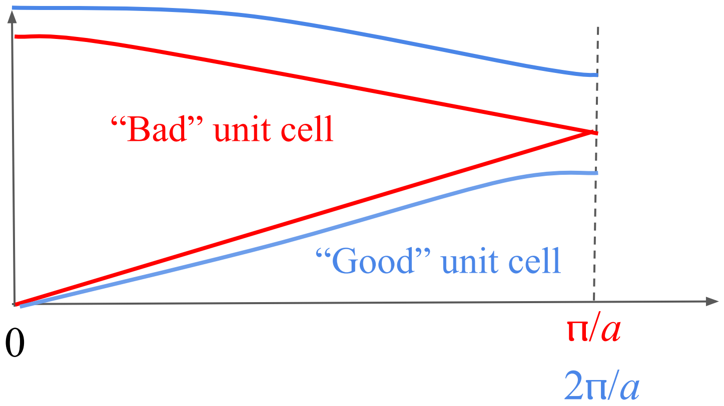

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At the Brillouin zone boundary we can choose the solution to Maxwell's equations as a real function. Smallest real matrix representation:

$$T_{a/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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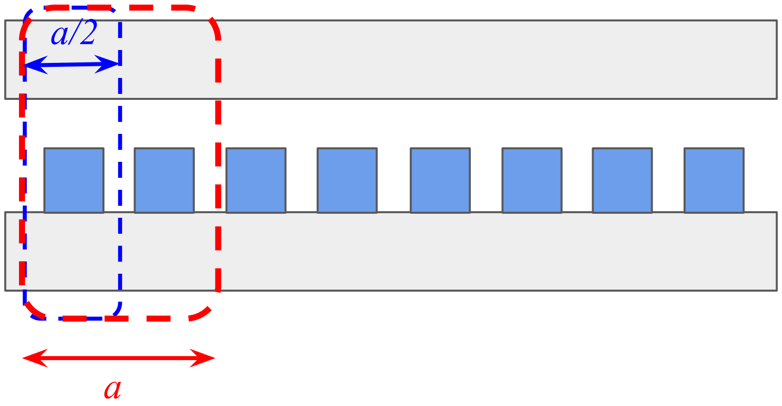

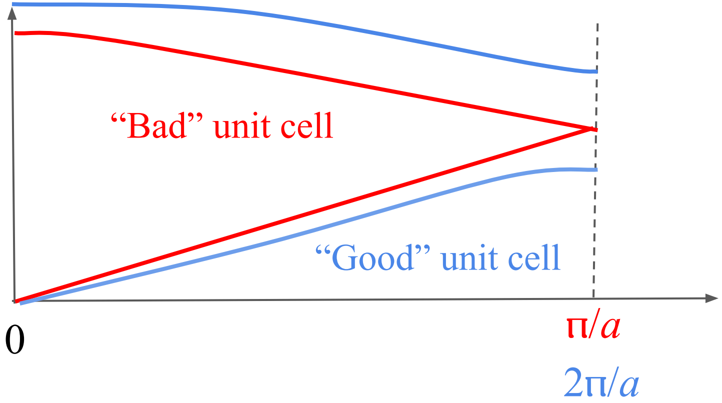

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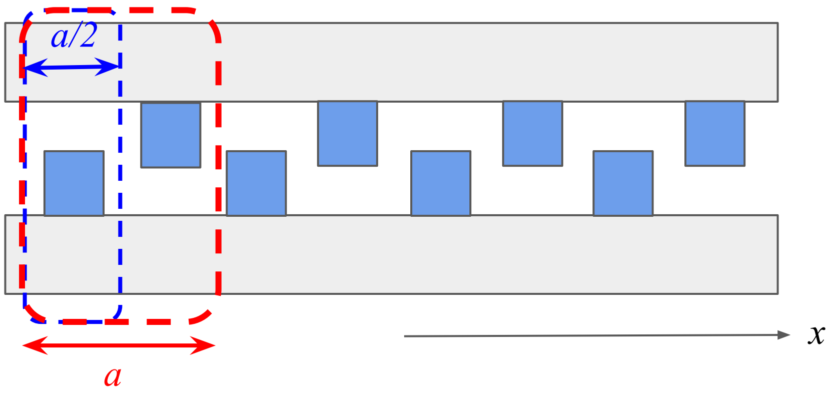
Two dimensional irreducible real representation.

Two bands meet at the Brillouin zone boundary

Degeneracy from glide and screw symmetry

Glide symmetry:



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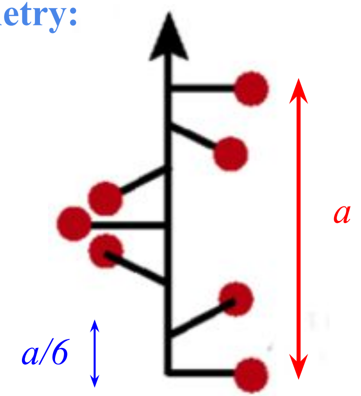
Screw symmetry:



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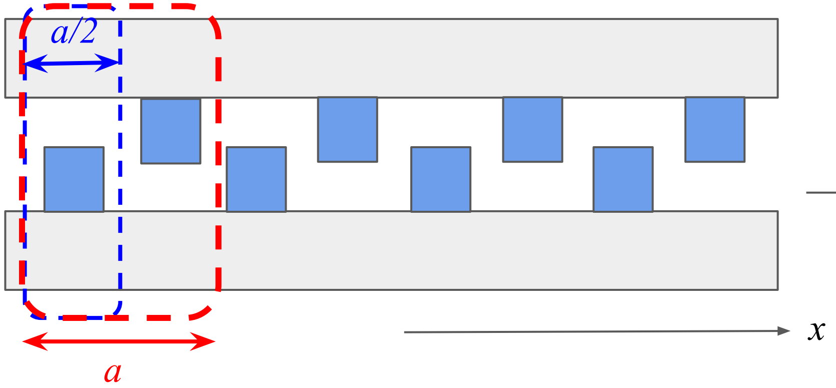

At the Brillouin zone boundary the wave satisfies,

$$E_z(x+a) = -E_z(x)$$

$$(T_{a/N})^N E_z = -E_z \quad \text{(n-fold screw axis)}$$

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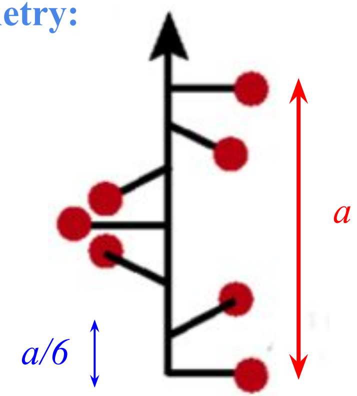

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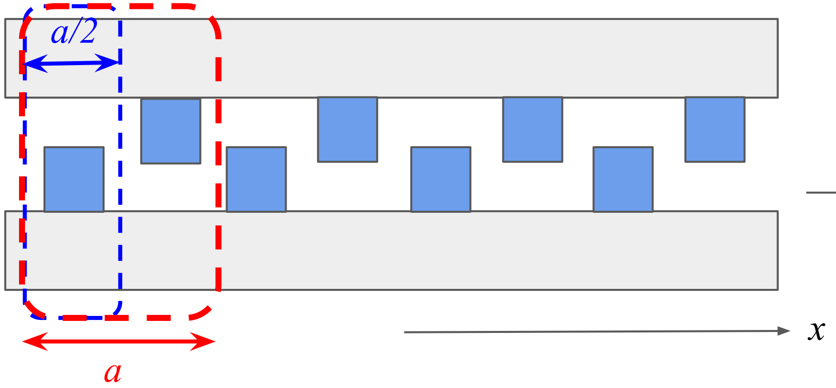

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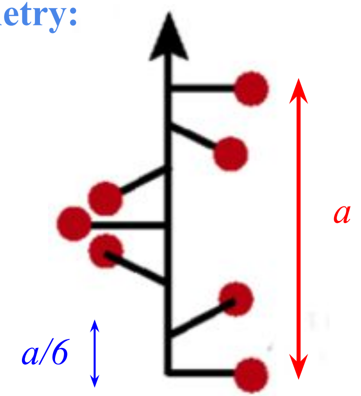

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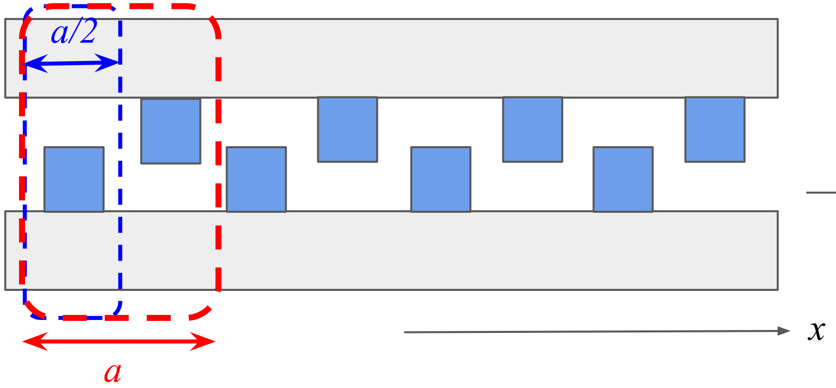
Smallest real matrix representation:

$$T_{a/N} = \begin{pmatrix} \cos(\pi/N) & \sin(\pi/N) \\ -\sin(\pi/N) & \cos(\pi/N) \end{pmatrix}$$

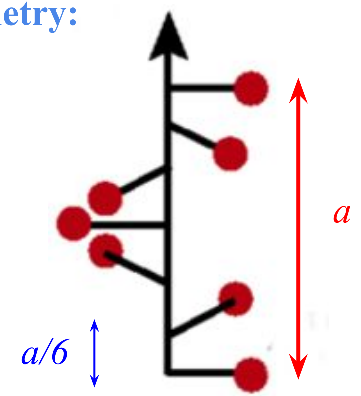
N=2 corresponds to glide symmetry transformation

Degeneracy from glide and screw symmetry

Glide symmetry:



Screw symmetry:



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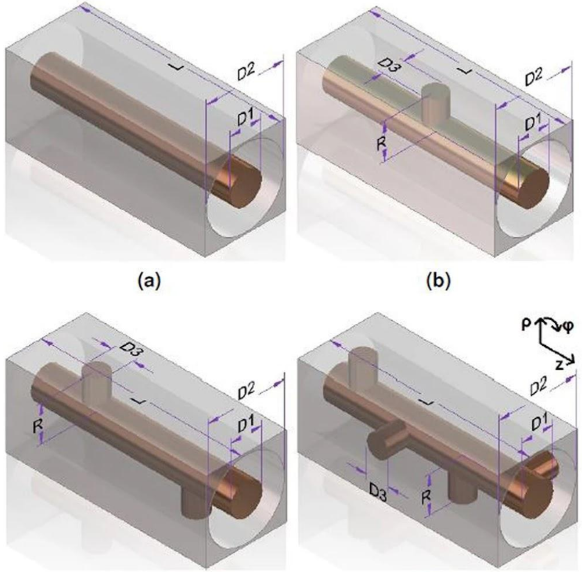
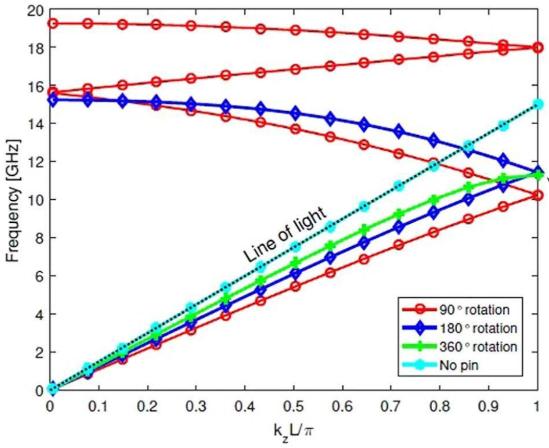


(a) (b) (c) (d)



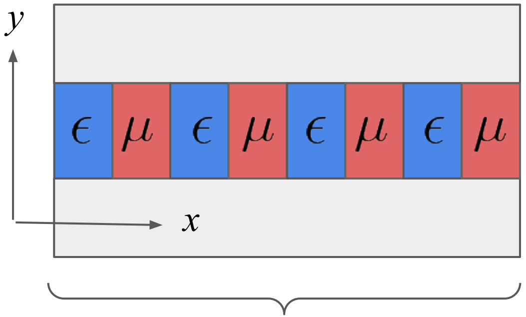
(e)

The order of the glide/screw axis makes no difference: always two bands meet.

O. Dahlberg, R. C. Mitchell-Thomas, & O. Quevedo-Teruel "*Reducing the Dispersion of Periodic Structures with Twist and Polar Glide Symmetries.*" *Sci Rep* **7,** 10136 (2017)

Generalized glide/screw symmetry

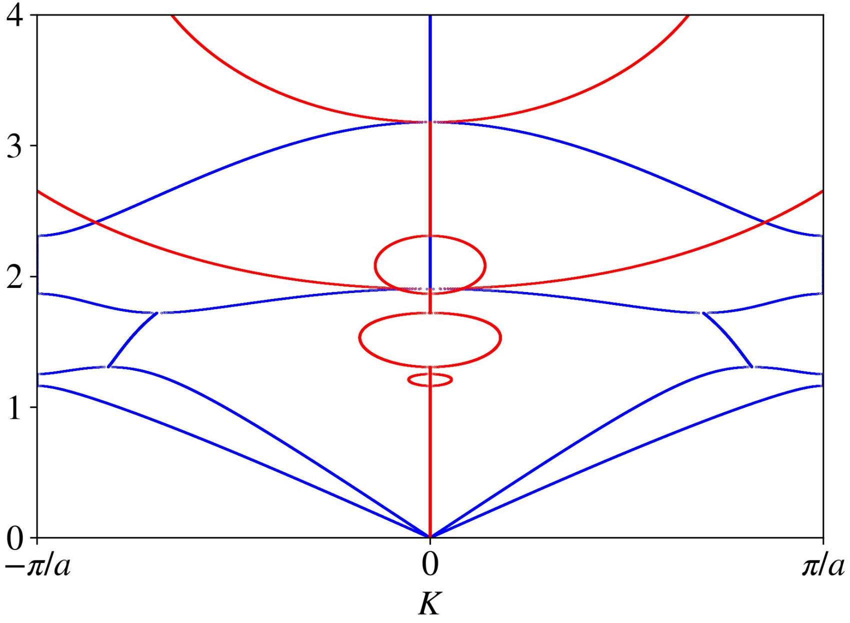
But the symmetry need not be geometrical. We could try other things...**like polarization**...



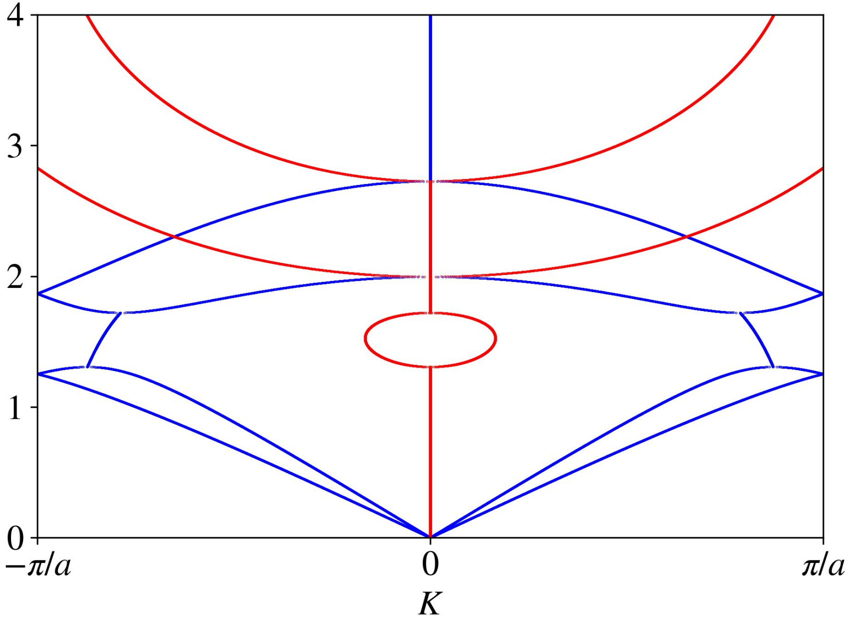
e.g. Alternating anisotropic electric and magnetic materials

Generalized glide/screw symmetry

Without “glide polarization” symmetry

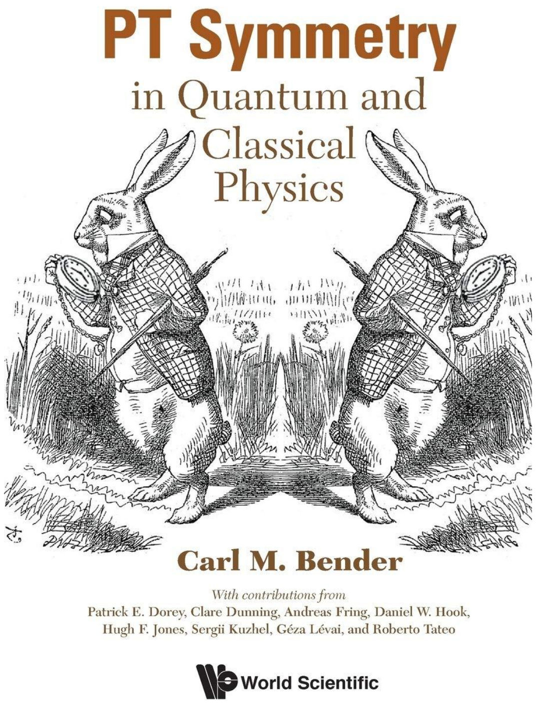


With “glide polarization” symmetry

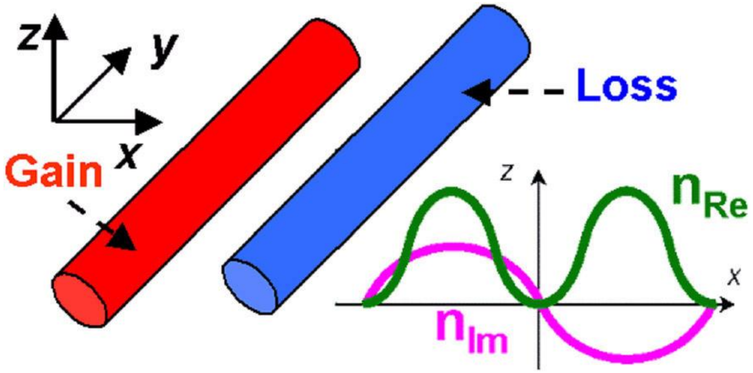


Generalized glide/screw symmetry

PT-symmetry (not quite ‘glide’ symmetry)



Electromagnetic structures need not only have spatial or polarization symmetries. **We can also have symmetries with respect to the direction of time.**



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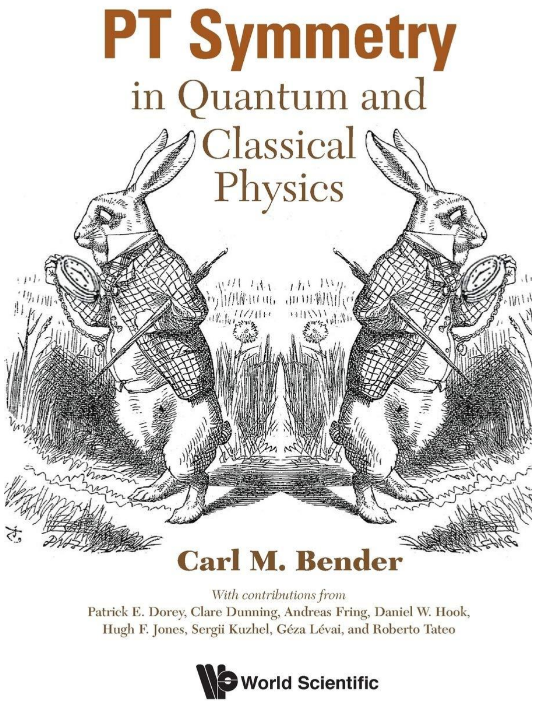

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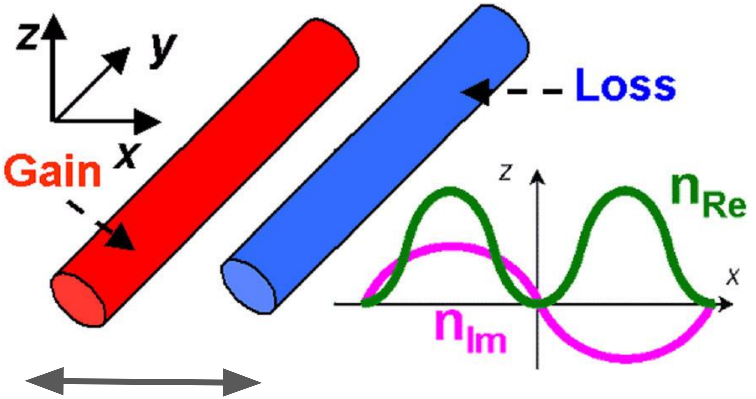
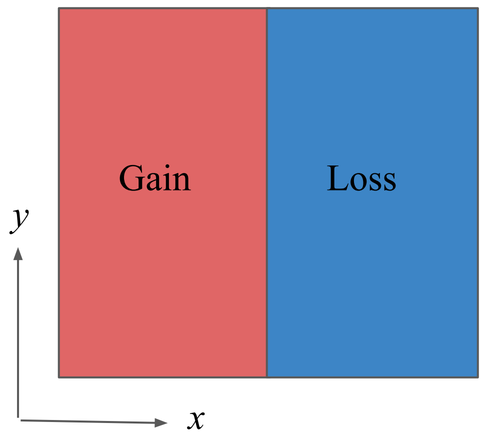


$$x' = -x$$

$$\epsilon' = \epsilon^{\star}$$

PT-symmetry (not quite ‘glide’ symmetry)



1D Electromagnetic Helmholtz equation:

$$\left[\frac{d^2}{dx^2} + k_0^2\epsilon(x) - k_z^2\right] E_z = 0$$

Perform transformation *x'*=-*x*, and complex conjugation

k_0 real: symmetry $E_z'(x) = E_z^\star(-x)$

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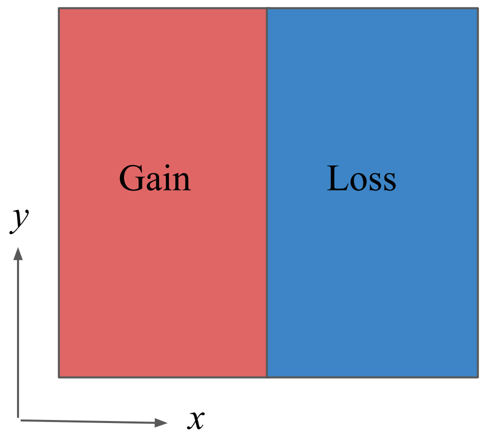

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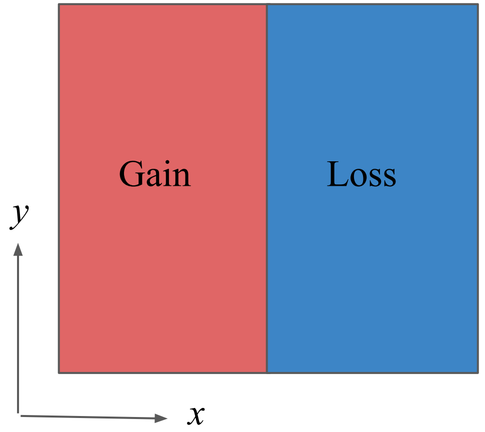

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k_0 real: symmetry $E_z'(x) = E_z^\star(-x)$

$$\longrightarrow T_{PT}E_z(x) = E_z^\star(-x)$$

$$\longrightarrow T_{PT}^2 = 1$$

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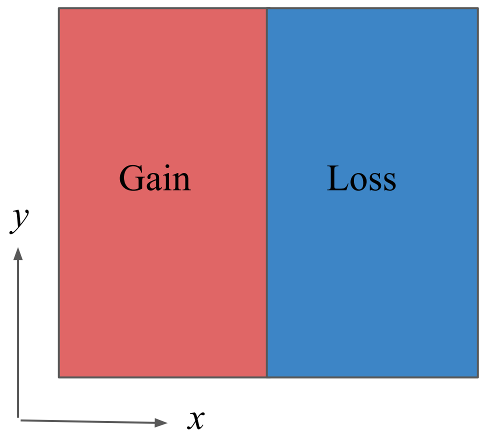

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$$\longrightarrow T_{PT}E_z(x) = E_z^\star(-x)$$

$$\longrightarrow T_{PT}^2 = 1$$

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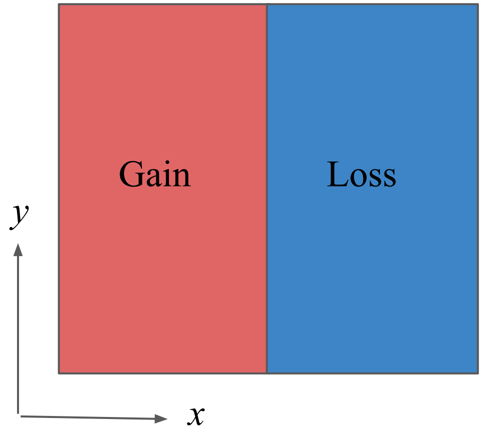
Very simple 1D irreducible representation!

PT-symmetry (not quite 'glide' symmetry)



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$$\left[\frac{d^2}{dx^2} + k_0^2\epsilon(x) - k_z^2\right] E_z = 0$$

Perform transformation *x'*=-*x*, and complex conjugation

PT-symmetric phase

k_0 real: symmetry $E_z'(x) = E_z^\star(-x)$

$\longrightarrow T_{PT}E_z(x) = E_z^\star(-x)$

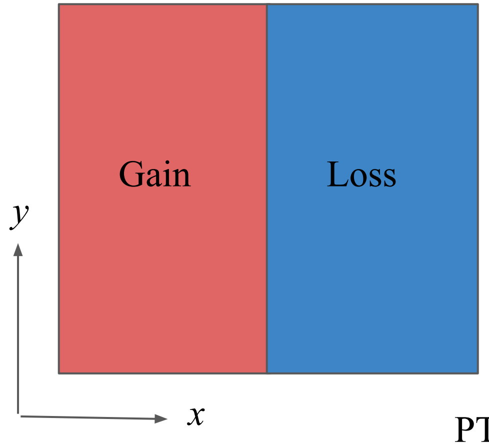
$\longrightarrow T_{PT}^2 = 1$

$\longrightarrow T_{PT} = \pm 1$

Very simple 1D irreducible representation!

$\longrightarrow$ **2 non-degenerate solutions, either even or odd**

PT-symmetry (not quite 'glide' symmetry)



1D Electromagnetic Helmholtz equation:

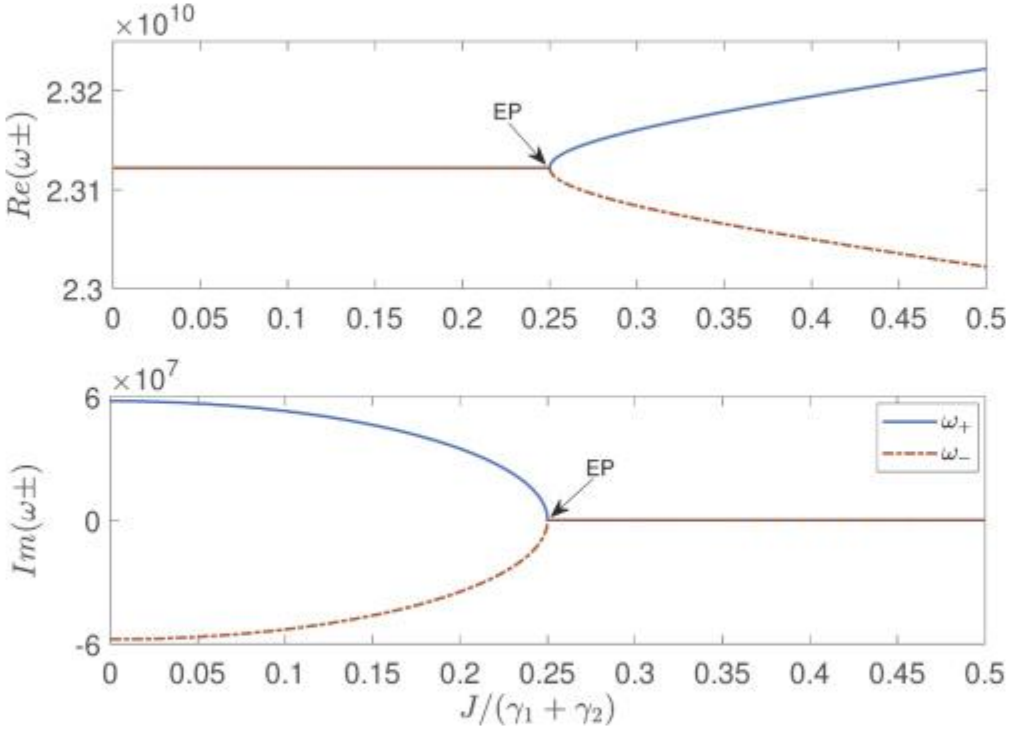
$$\left[\frac{d^2}{dx^2} + k_0^2\epsilon(x) - k_z^2\right] E_z = 0$$

Perform transformation *x'*=-*x*, and complex conjugation

PT-broken phase $\left\{ \right.$ k_0 complex: $E_z(x)$ & $E_z^\star(-x)$ are now solutions to different equations , with frequencies k_0 and k_0*

If some parameter (e.g. the scale of the permittivity or wavevector k_z) is changed, the PT-symmetric phase can become broken. This is called the ***exceptional point***.

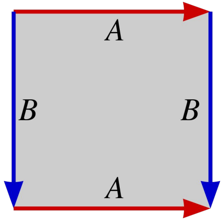
PT-symmetry (not quite ‘glide’ symmetry)

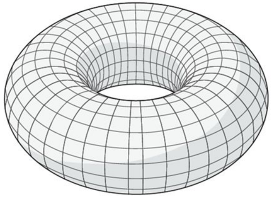


Higher symmetries and topology

Visualizing higher symmetries:

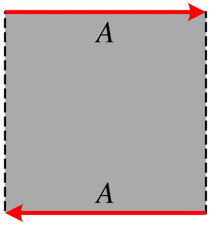
Ordinary periodic medium

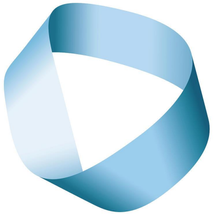




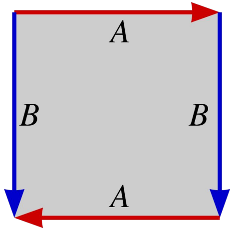
Torus

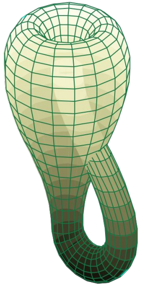
HIgher symmetry along one axis





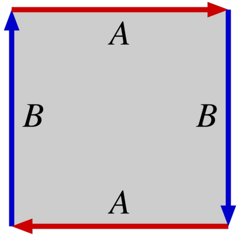
Mobius strip

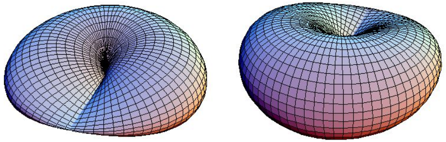




Klein bottle

Higher symmetry along two axes





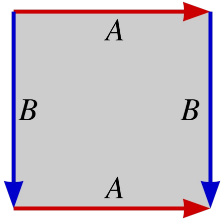
Real projective plane

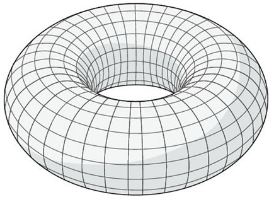
Higher symmetries and topology

Visualizing higher symmetries:

c.f. tomorrow's lecture on topology

Ordinary periodic medium





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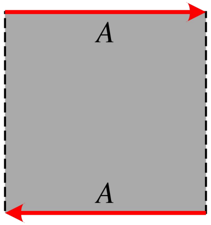

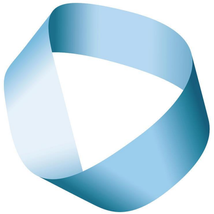

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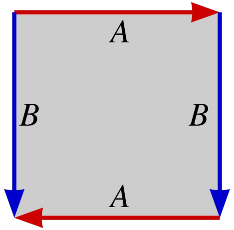

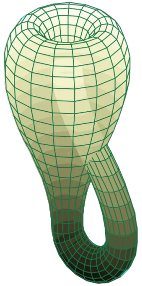

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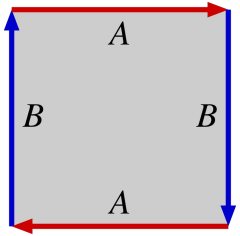

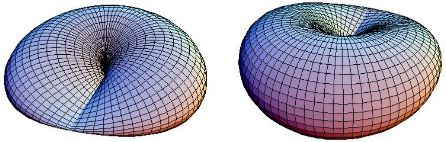

Real projective plane

Summary

- If you are interested in symmetry, **group theory is a useful thing to know!**
- Groups describe symmetries, and their **irreducible representations tell us about the number of resulting degeneracies.**
- There is an important formula relating the number of elements of a group, and the dimensions of the irreducible representations

$$\sum_\mu n_\mu^2 = g$$

- **We can use group theory to derive Bloch's theorem and degeneracies at the Brillouin zone boundary** in the case of a badly chosen unit cell, glide symmetry, screw symmetry, and generalized glide symmetry.
- There seem to be many possible higher symmetry electromagnetic structures (the analogue of hexagonal ice?). **Group theory is a useful tool to explore this space!**