What happens when two bands meet?: topology and electromagnetism

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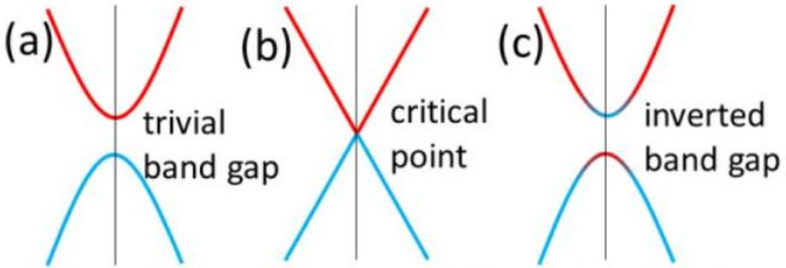
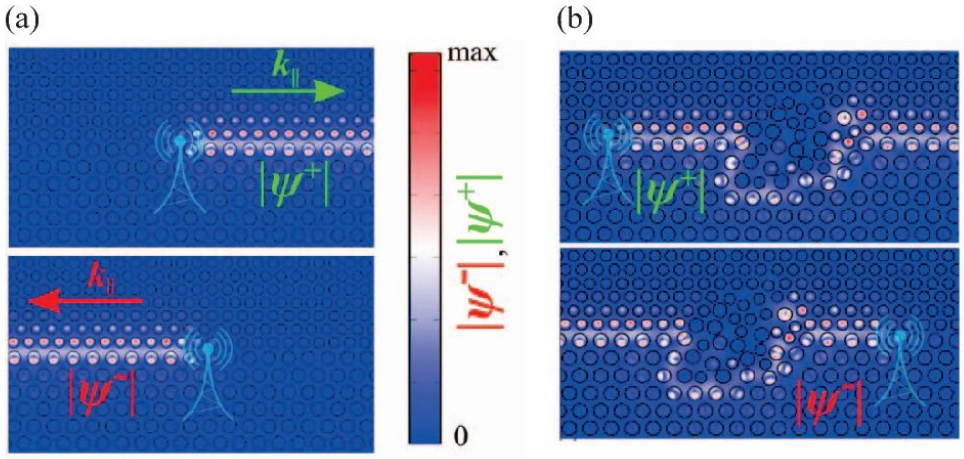
Topology in wave physics

A ***topological invariant*** (Chern number) can be assigned to a pass band. This number can only be changed through closing the gap.

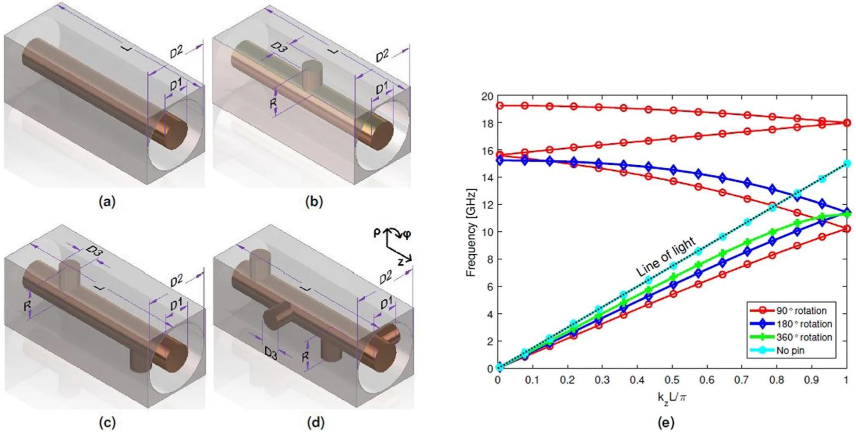


The difference in the topological invariant between two materials determines the number of ***edge states*** that their interface can support.

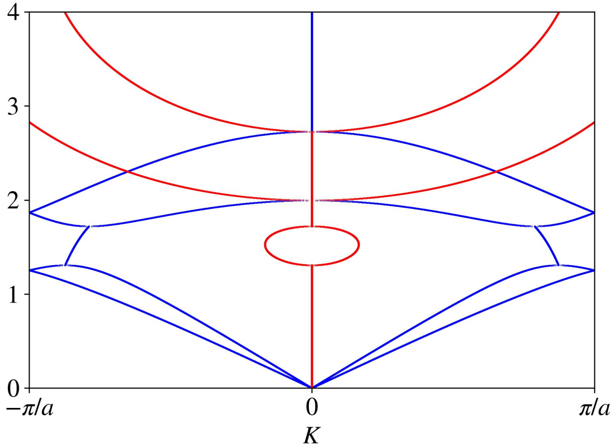


Topology in wave physics

We can use "higher symmetries" to close a band gap, and then invert the band



O. Dahlberg, R. C. Mitchell-Thomas, & O. Quevedo-Teruel
"*Reducing the Dispersion of Periodic Structures with Twist and Polar Glide Symmetries.*" *Sci Rep* **7,** 10136 (2017)



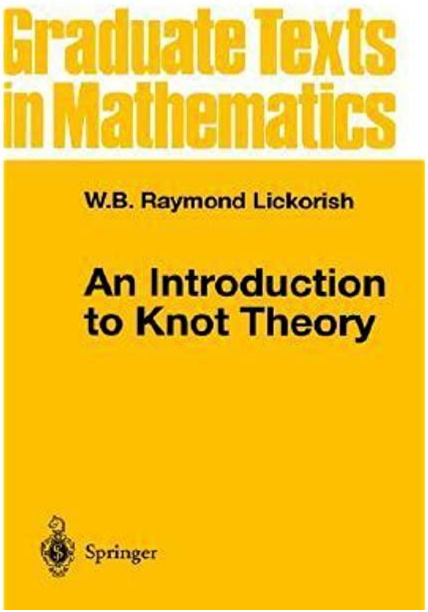
Band closure using "glide polarization" symmetry

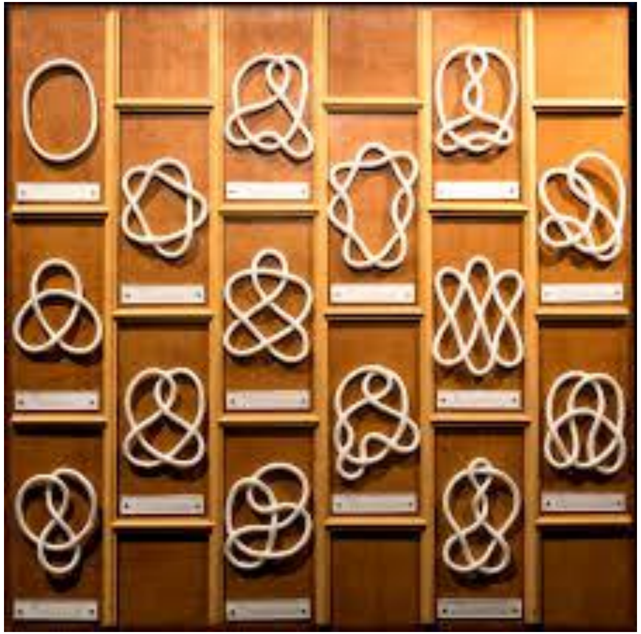
Part 1: A sketch of differential topology

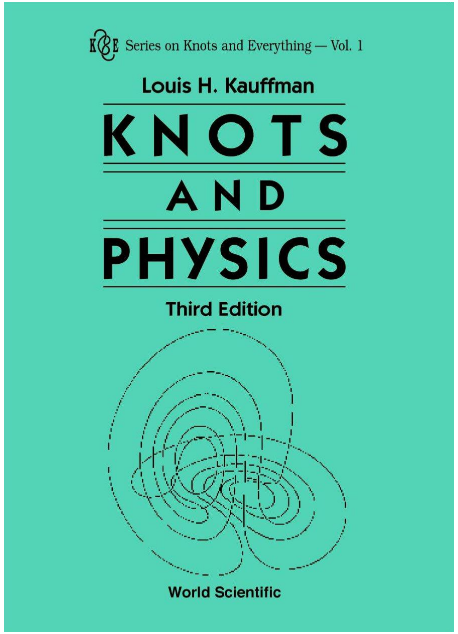
What is topology all about?

Topology is about categorizing objects on the basis of their most basic structure, e.g. how they are connected.

Topology can be used to classify the connectedness of loops of string: topologically distinct knots can only be transformed into one another through cutting the string.







Examples of knots

What is topology all about?

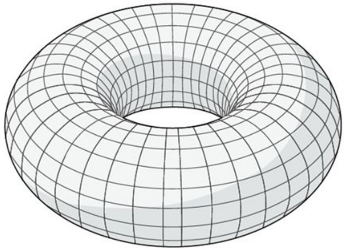
Topology is about categorizing objects on the basis of their most basic structure, e.g. how they are connected.

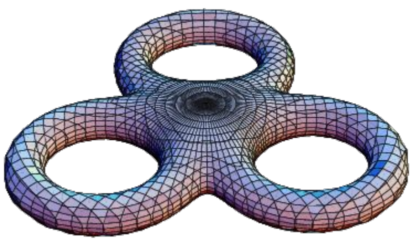
Topology can also be used to classify 2D surfaces: Topologically distinct surfaces can only be transformed into each other through breaking open the surface.

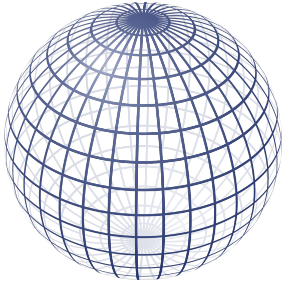
A moray eel ~ a torus

A pretzel ~ a three torus

Morph~ a sphere







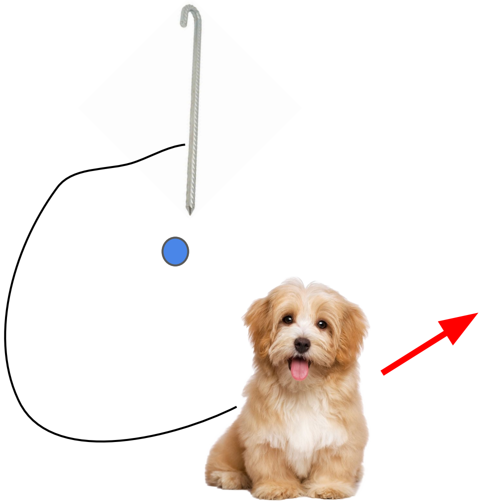
1D objects - winding numbers

Topology of paths - winding numbers:

e.g. a telegraph pole

1D objects - winding numbers

Topology of paths - winding numbers:



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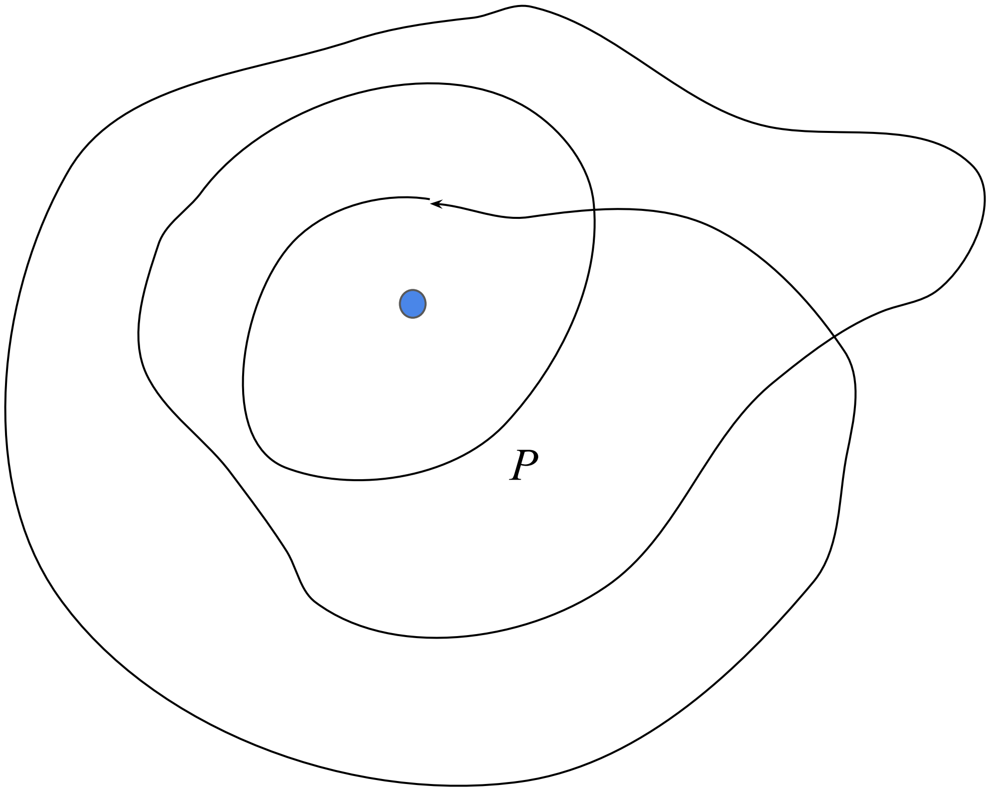


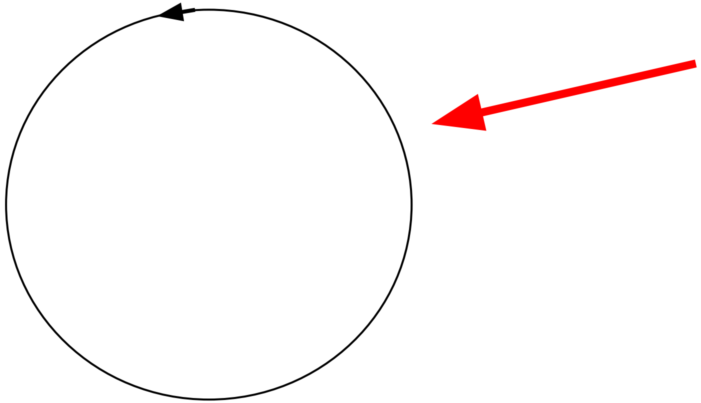
1D objects - winding numbers

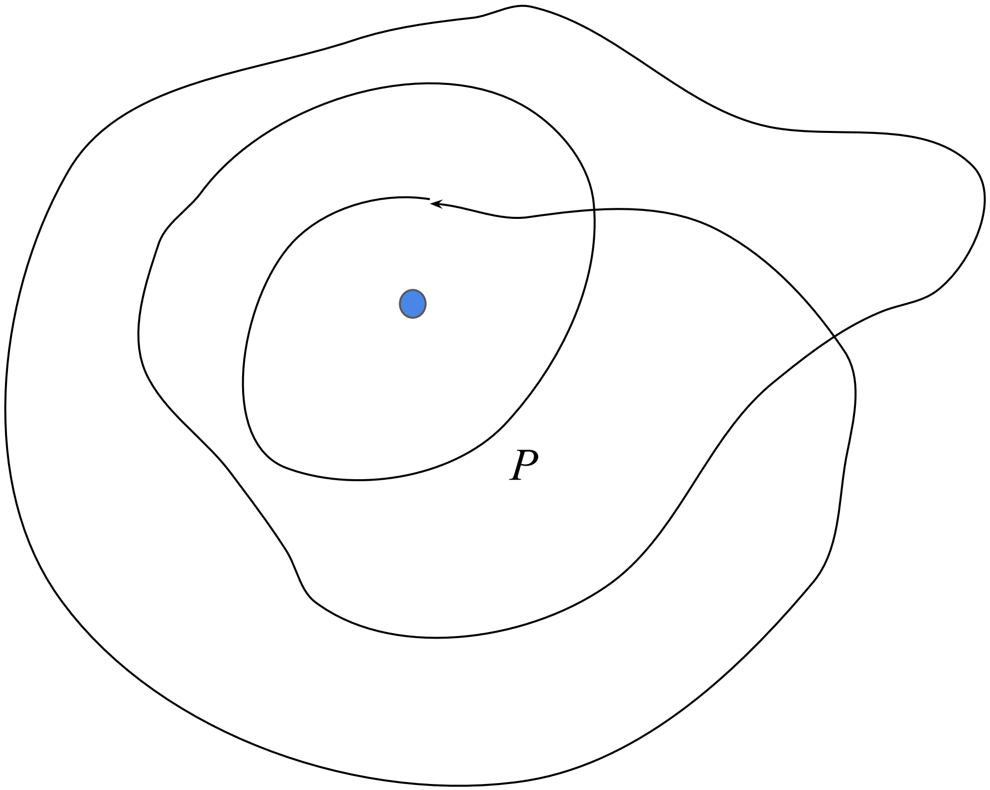
Topology of paths - winding numbers:

Winding number is a topological invariant.

$$\boxed{n = \frac{1}{2\pi}\int_P d\theta}$$

How many times do we move round the circle?





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$$= \frac{\hat{\boldsymbol{z}} \times \boldsymbol{x}}{x^2 + y^2} \cdot d\boldsymbol{x}$$

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$$\longrightarrow n = \oint_P \underbrace{\frac{\hat{\boldsymbol{z}} \times \boldsymbol{x}}{2\pi|\boldsymbol{x}|^2}} \cdot\, d\boldsymbol{x}$$

Vector potential $\boldsymbol{A}$ for a line of flux along

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Applying Stokes' theorem

$$\longrightarrow n = \oint_P \boldsymbol{A} \cdot d\boldsymbol{x} = \int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S}$$

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$$\longrightarrow n = \underbrace{\int_S dx\,dy\,\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{x}}{2\pi|\boldsymbol{x}|^2}\right)}$$

Counts number of "magnetic flux lines" passing through *S*.

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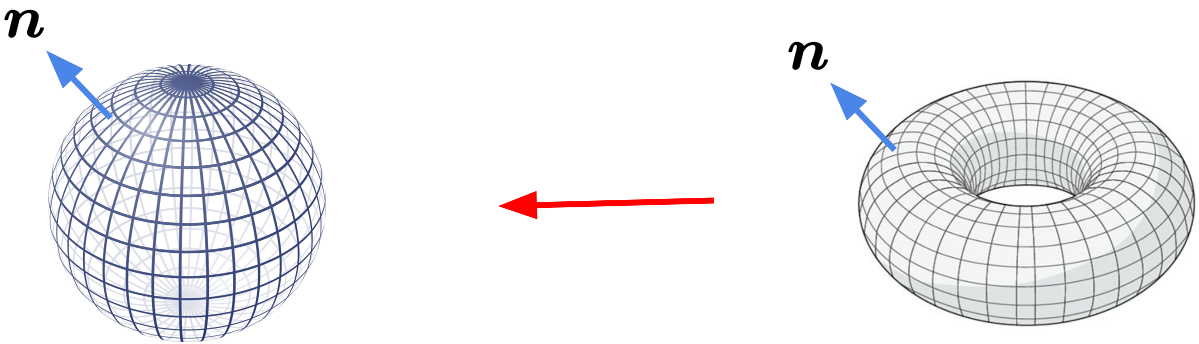
Integer = integral over S, which can be continuously deformed

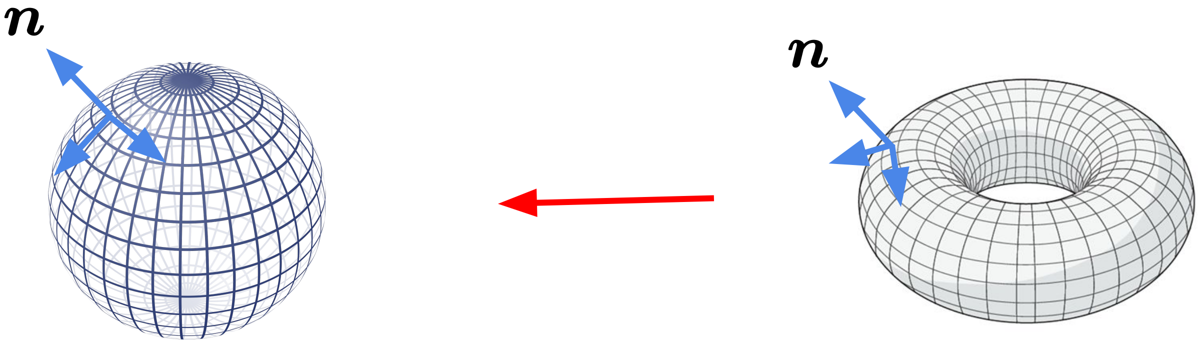
2D surfaces - the Gauss-Bonnet theorem

Topology of surfaces - 2D winding numbers: For any given surface, how many times does the surface normal cover all possible directions?



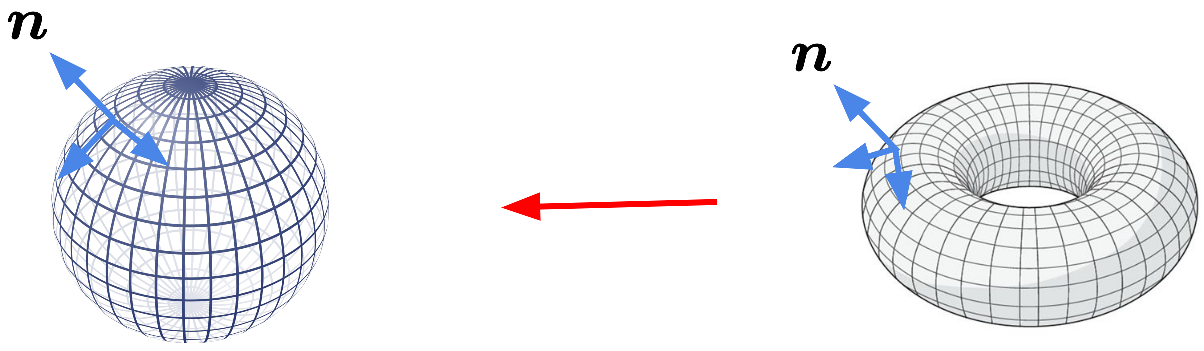
How many times does $\boldsymbol{n}$ wrap around the sphere?

2D surfaces - the Gauss-Bonnet theorem



$$dS = \sin(\theta)\, d\theta\, d\phi = \boldsymbol{n} \cdot \frac{\partial \boldsymbol{n}}{\partial \theta} \times \frac{\partial \boldsymbol{n}}{\partial \phi} d\theta\, d\phi$$

2D surfaces - the Gauss-Bonnet theorem



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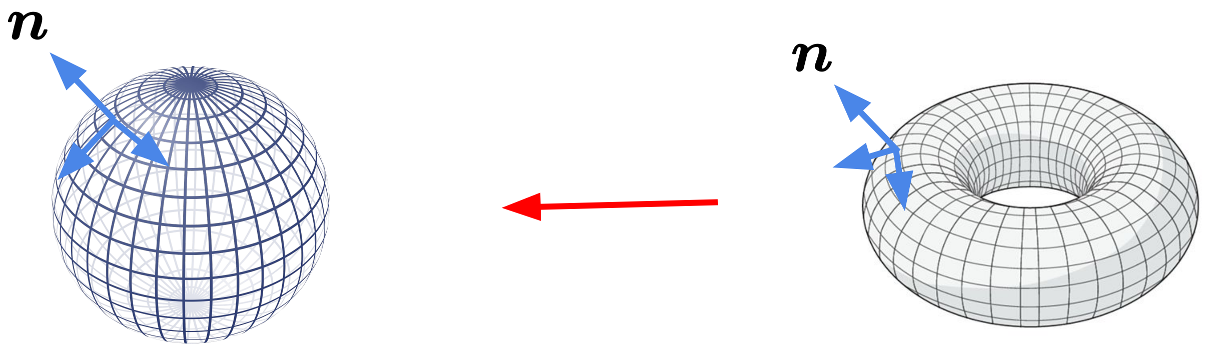
2D surfaces - the Gauss-Bonnet theorem



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Winding number: $$N = \frac{1}{4\pi} \int_S \boldsymbol{n} \cdot \frac{\partial \boldsymbol{n}}{\partial \xi} \times \frac{\partial \boldsymbol{n}}{\partial \zeta} d\xi\, d\zeta$$

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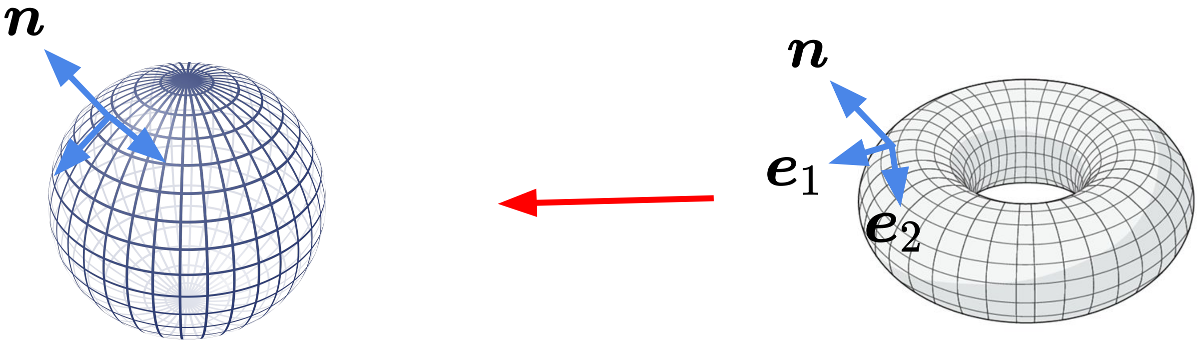

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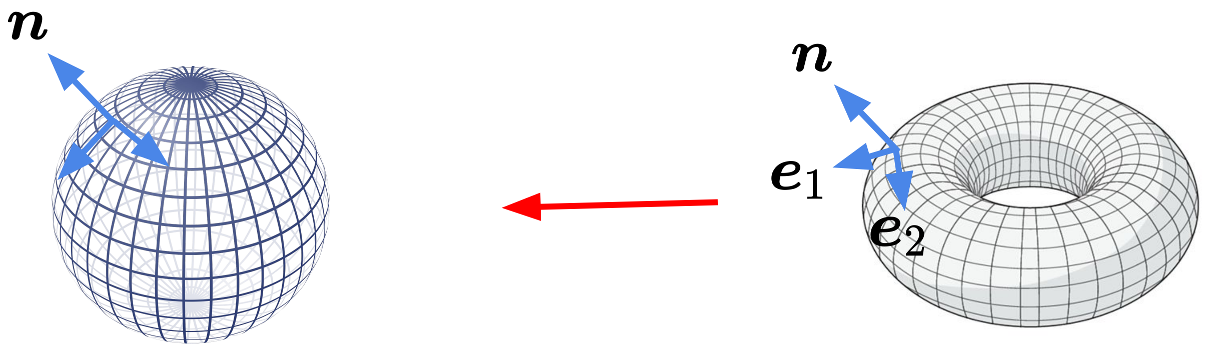
$$\longrightarrow \quad \frac{\partial\boldsymbol{n}}{\partial\xi} = \frac{\boldsymbol{e}_1}{R_1}$$

$$\longrightarrow \quad \frac{\partial\boldsymbol{n}}{\partial\zeta} = \frac{\boldsymbol{e}_2}{R_2}$$

2D surfaces - the Gauss-Bonnet theorem



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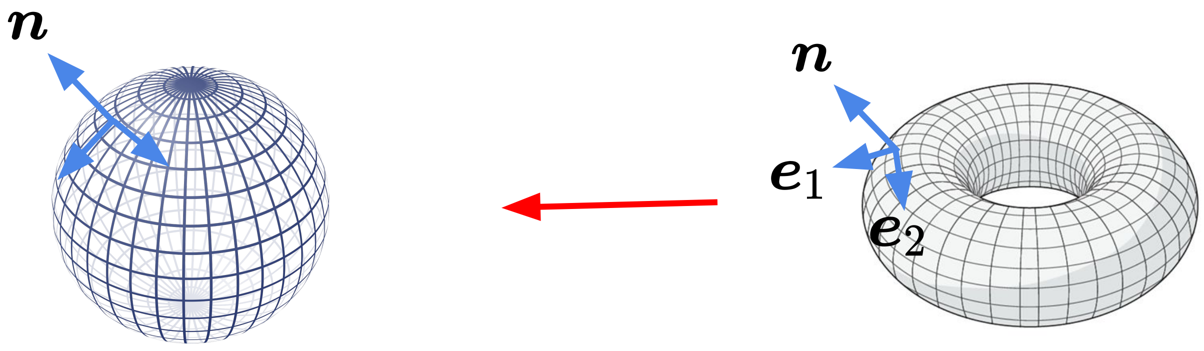
2D surfaces - the Gauss-Bonnet theorem



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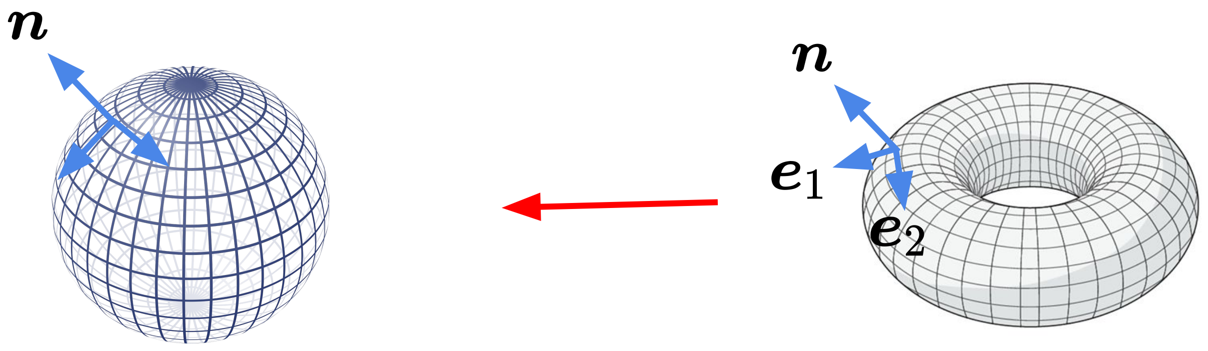
The Gaussian curvature, R

2D surfaces - the Gauss-Bonnet theorem



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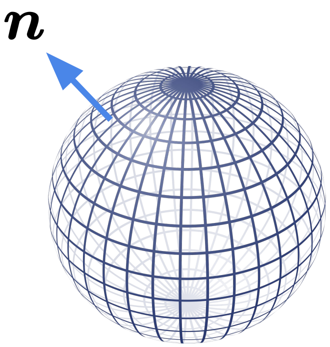
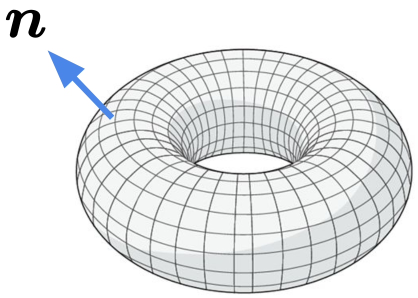
Half the Euler characteristic, □

The Gaussian curvature, *R*

2D surfaces - the Gauss-Bonnet theorem





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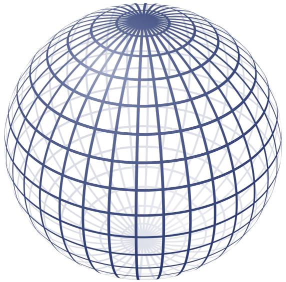
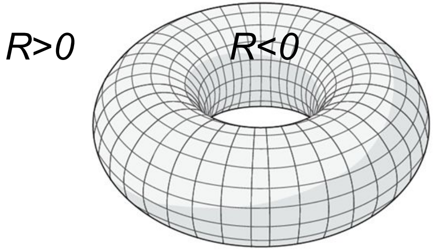
The Gauss-Bonnet theorem: $$\chi = \frac{1}{2\pi}\int_S R\,dA$$

The Gauss-Bonnet theorem: examples



$\chi = 2$

Although the surface (and hence R) is continuously deformable, the Euler characteristic remains unchanged unless the surface is broken open.



$\chi = 0$

The Gauss-Bonnet theorem: $\chi = \frac{1}{2\pi} \int_S R \, dA$

$\chi = -2$

Every additional hole in the closed surface adds one winding around the sphere in a negative sense, hence reducing the Euler characteristic by two.

Another way to write the Gauss-Bonnet theorem

There is another way to write the Gauss-Bonnet theorem, that directly connects with our earlier discussion of the 1D winding number.

Gauss-Bonnet theorem:
(Winding on the sphere)

$$N = \frac{1}{4\pi} \int_S \boldsymbol{n} \cdot \frac{\partial \boldsymbol{n}}{\partial \xi} \times \frac{\partial \boldsymbol{n}}{\partial \zeta} d\xi \, d\zeta$$

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Writing:

$$x_1 = \xi$$

$$x_2 = \zeta$$

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$$= \frac{1}{4\pi}\int_S \left[\frac{\partial \boldsymbol{e}_1}{\partial x_1}\cdot\frac{\partial \boldsymbol{e}_2}{\partial x_2} - \frac{\partial \boldsymbol{e}_1}{\partial x_2}\cdot\frac{\partial \boldsymbol{e}_2}{\partial x_1}\right] dx_1\,dx_2$$

where:

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Gauss Bonnet theorem in terms of 'magnetic flux'

Another way to write the Gauss-Bonnet theorem

But how is this supposed to make sense? I thought there was a thing called Stokes' theorem!

If $\quad \chi = \dfrac{1}{2\pi} \displaystyle\int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d^2\boldsymbol{x}\,,$ then

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$$\longrightarrow \quad \int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d^2\boldsymbol{x} = \oint \boldsymbol{A} \cdot d\boldsymbol{l} \stackrel{?}{=} 0\,!$$

Where has the topology gone?

https://www.youtube.com/watch?v=HsRFlHlS2Cs

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Where has the topology gone?

We have discovered the ***"Hairy ball theorem"***:

The ability to comb hair tangent to a surface without "defects" is governed by the value of the Euler characteristic.

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The Gauss-Bonnet theorem and the Berry connection

$$\chi = \frac{1}{2\pi}\int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d^2\boldsymbol{x} \qquad \text{where:} \qquad A_i = \boldsymbol{e}_1 \cdot \frac{\partial \boldsymbol{e}_2}{\partial x_i}$$

The Gauss-Bonnet theorem and the Berry connection

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With $\boldsymbol{A}_i$ defined as above, the defects in $\boldsymbol{A}_i$ record the topology of the surface.

We can write it in a simpler form if we define a complex vector: $\boldsymbol{e} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2)$

The Gauss-Bonnet theorem and the Berry connection

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The Berry connection for a two-component wavefunction!

But we needn't use two component vectors!

The Gauss-Bonnet theorem and the Berry connection

Generalization of the Gauss-Bonnet theorem:

$$n = \frac{1}{2\pi}\int_S \boldsymbol{\nabla}\times\boldsymbol{A}\cdot d^2\boldsymbol{x} \qquad \text{where:} \qquad A_i = -\mathrm{i}\langle\psi|\frac{\partial}{\partial x_i}|\psi\rangle$$

n is the "***Chern number***" (integral of the first Chern class), ***equal to the Euler characteristic when ψ is a two component object.***

What does n ***measure?*** In general ***it tells us whether two different complex vector fields*** $|\psi\rangle$ ***can be smoothly deformed into each other.*** When the vector is two dimensional this also tells us about the topology of the surface.

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For anyone interested: this can be generalized to higher dimensional surfaces (***higher Chern classes***), and influences the spectrum of waves on the surface (***Atiyah-Singer index theorem***).

5 minute break!

The story so far...

- Topology is a subject concerned with classifying objects. ***If two objects have different topological invariants then they cannot be smoothly deformed into one another.***

- ***For line-like objects we can define the winding number around a point.*** It is possible to re-write this in terms of the effective magnetic flux through the surface enclosed by the line.

- For 2D surfaces we can define a "winding number" as the number of times a surface normal wraps around the sphere. ***This is half the Euler characteristic***.

- The ***Gauss-Bonnet theorem relates the Euler characteristic to the integral of the curvature***.

$$\chi = \frac{1}{2\pi} \int_S R\, dA$$

- This can be generalized to the "***Chern number***", where the curvature is replaced by the curl of the Berry connection

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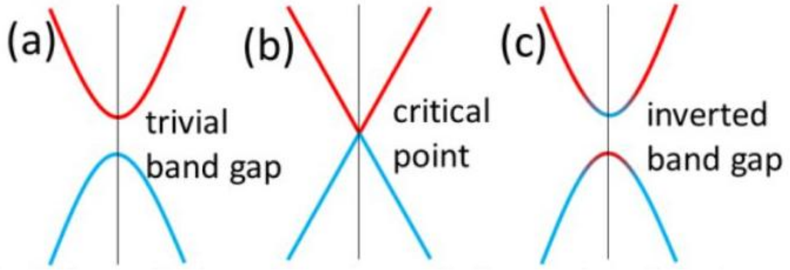
Part 2: Topology and band structure

Topology and band structure

The idea: Bands can be classified in terms of their Chern number:



Suppose we have only two bands

Topology and band structure

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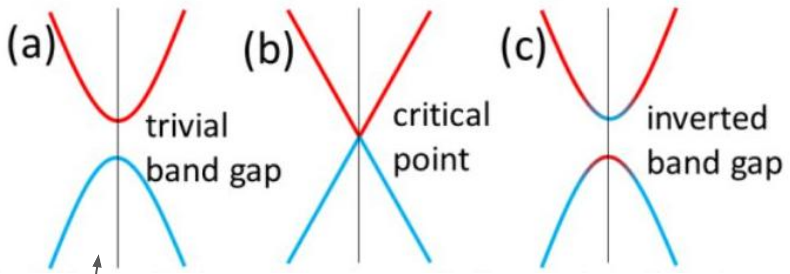
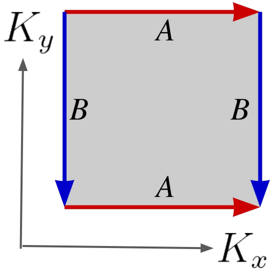


Suppose we have only two bands

Reciprocal space is periodic



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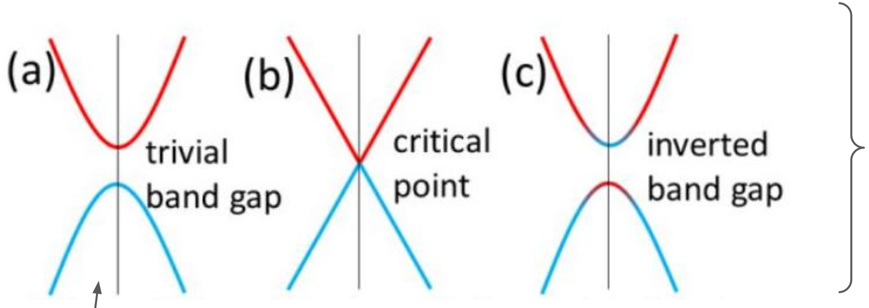

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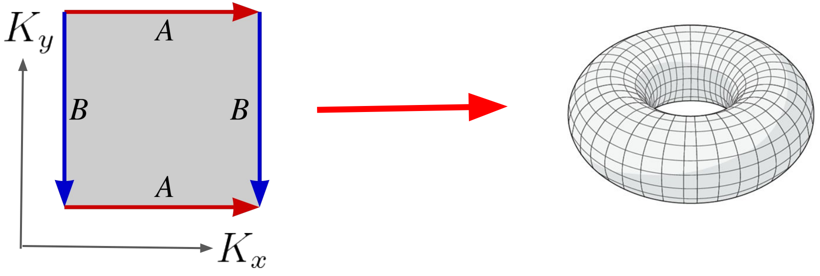

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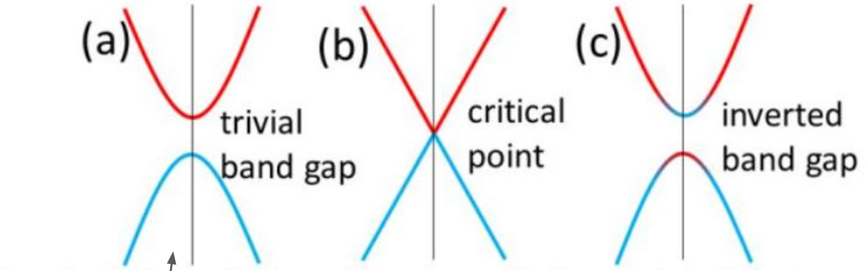

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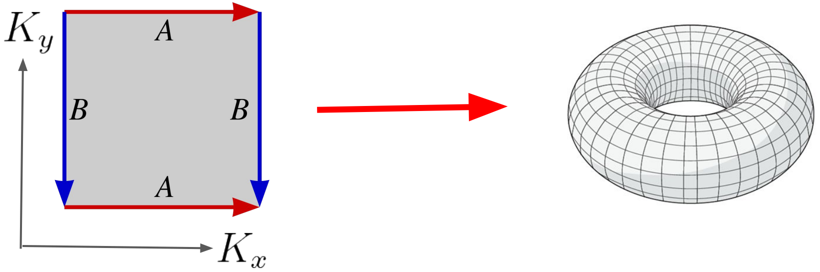

Each point on the torus is labelled by a Bloch vector, $\boldsymbol{K}$.

At each point there is a vector $|\psi_{\boldsymbol{K}}\rangle$, which is e.g. formed from the Bloch function at every point in the unit cell.

Topology and band structure

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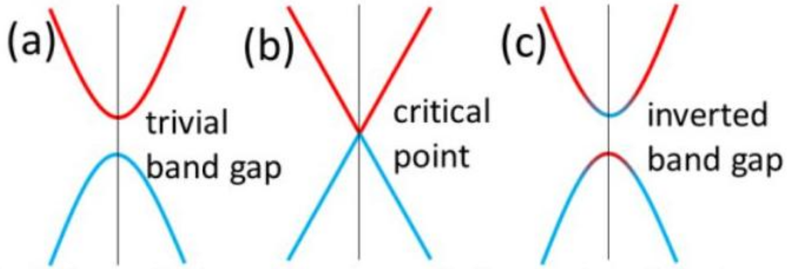

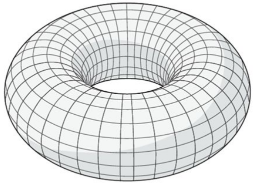

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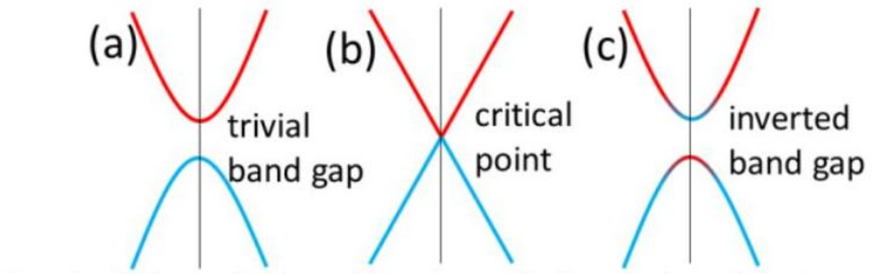

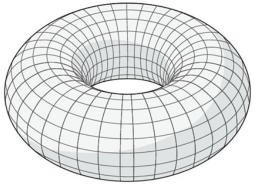

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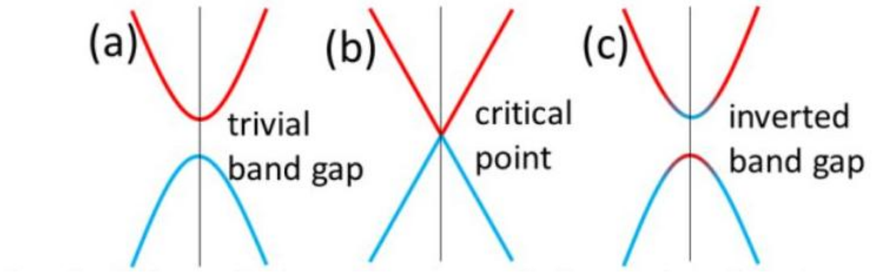
$$\boxed{n = \frac{1}{2\pi}\int_{B.Z.} \boldsymbol{\nabla}_{\boldsymbol{K}} \times \boldsymbol{A}_{\boldsymbol{K}} \cdot d^2\boldsymbol{K} \quad \text{where,} \quad \boldsymbol{A}_{\boldsymbol{K}} = -\mathrm{i}\langle\psi_{\boldsymbol{K}}|\boldsymbol{\nabla}_{\boldsymbol{K}}|\psi_{\boldsymbol{K}}\rangle}$$

Explicitly: $\boldsymbol{A}_{\boldsymbol{K}} = -\mathrm{i}\int d^2\boldsymbol{x}\, u^\star_{\boldsymbol{K}}(\boldsymbol{x})\boldsymbol{\nabla}_{\boldsymbol{K}}\, u_{\boldsymbol{K}}(\boldsymbol{x})$ (a big dot product!)

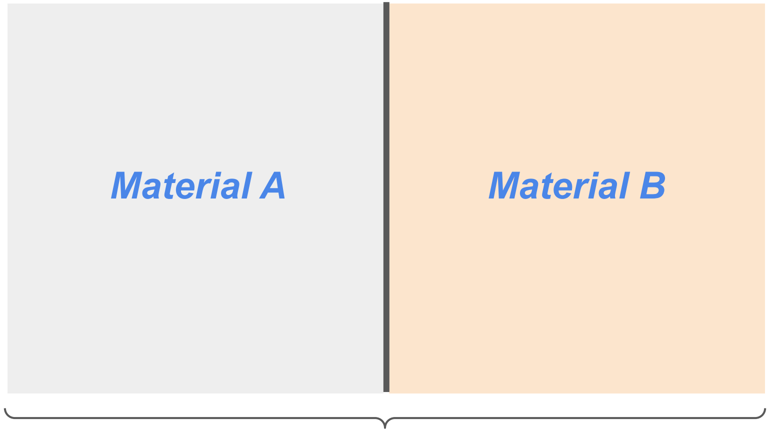
Topology and band structure

The idea: Suppose we have an interface between two materials with different Chern number:



n=0
Material A

n=1
Material B



Suppose material A is slowly changed into material B so that locally there is band structure.

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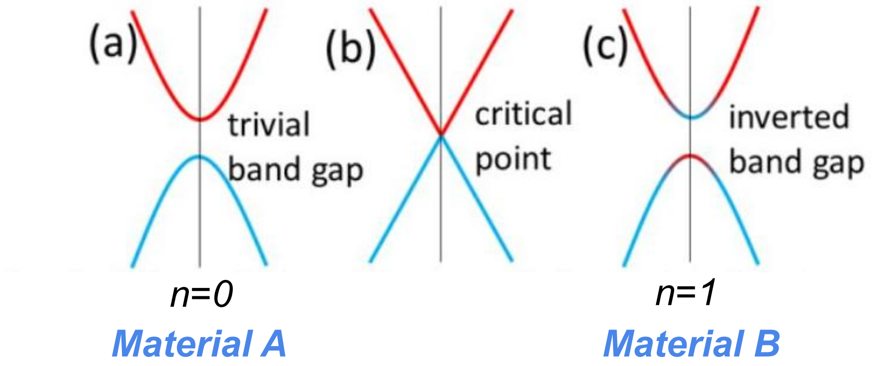

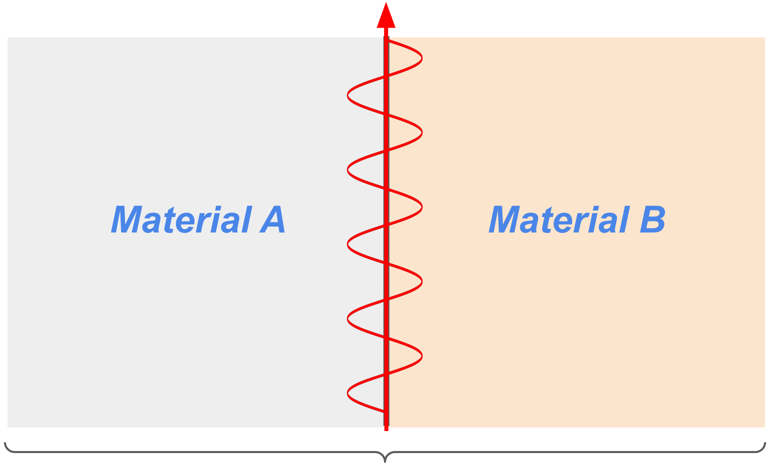

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Topology tells us that whatever we do to the grading, ***somewhere the band must close***, and then waves can propagate. ***These waves are edge states.***

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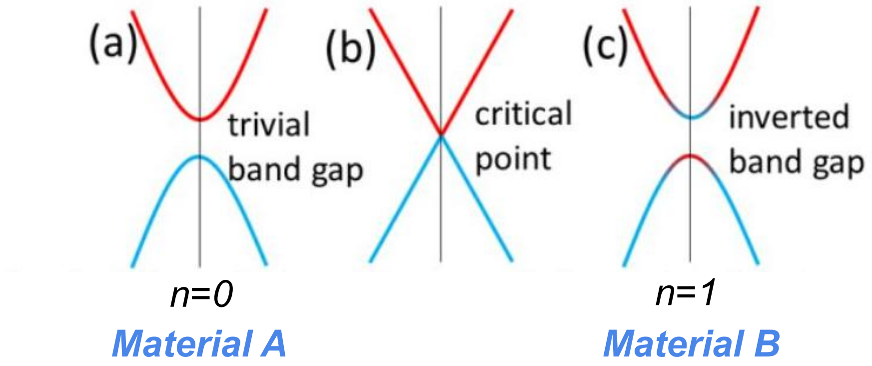

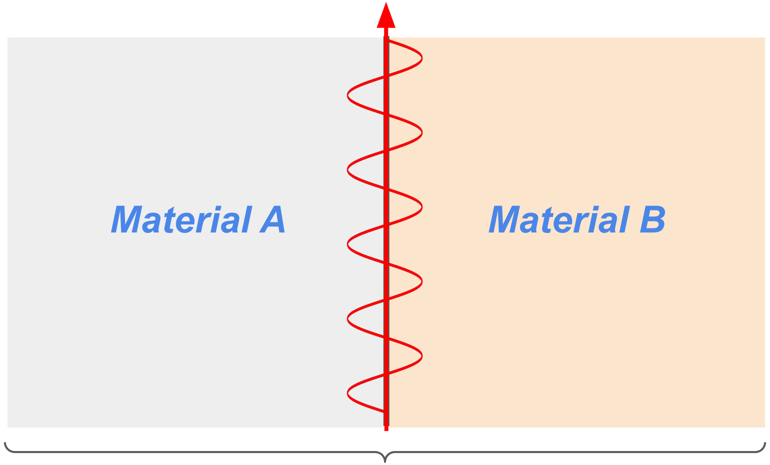

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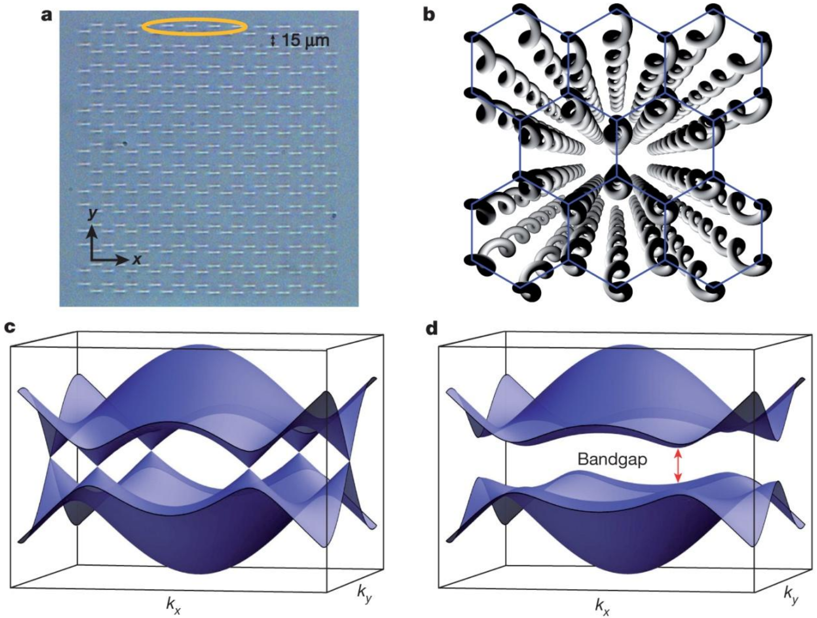
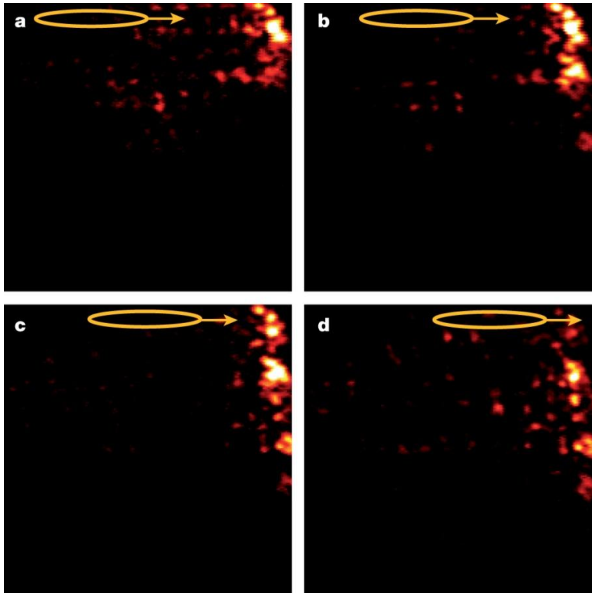
The Chern number counts the net number of defects on the torus, each of which could be undone separately. Therefore ***the net difference in Chern number counts the net number of edge states.***

Example





M. C. Rechtsman *et al. Nature* **496**, 196-200 (2013)

Summary

- Topology is a subject concerned with classifying objects. ***If two objects have different topological invariants then they cannot be smoothly deformed into one another.***

- The ***Gauss-Bonnet theorem relates the Euler characteristic to the integral of the curvature***.

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- We can define a Berry connection and Chern number for a pass band in a periodic medium. The difference in Chern numbers between media that share a band gap tells us how many states can be trapped at their interface.